Integration in $\mathbb{R}^m$

Let $f$ be a function on a topological space $X$. Let $U = \{ U \text{ open } X, f|_U = 0 \}$ be the family of all open sets on which $f$ restricts to 0.

Then the union of all elements in $U$ is in $U$, i.e., $U$ contains a largest element with respect to the partial ordering given by inclusion.

Therefore, there exists the largest open net $V$
in $X$ such that $f|_\mathcal{V} = 0$.  

The complement $X \setminus V$ is a closed set, which we call the support of $f$ and denote by $\text{supp} f$.

Let $C_0(\mathbb{R}^n)$ be the set of all continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support. Clearly, it is a real vector space. Moreover, with pointwise multiplication, it is an algebra.
We want to define a linear form on $C_c(\mathbb{R}^n)$
\[
   f \mapsto \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n.
\]
- the integral of $f$.

An $n$-cell $I^n$ is a product
\[
   [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]
\]
of closed intervals $[a_i, b_i], \ 1 \leq i \leq n$, i.e.
\[
   I^n = \{ (x_1, x_2, \ldots, x_n) \mid a_i \leq x_i \leq b_i, \ 1 \leq i \leq n \}.
\]
Since support of $f$ is compact, it is a bounded set in $\mathbb{R}^n$ and contained in some $n$-cell $I^n$. 
So we shall define

\[ \int_{\mathbb{R}^n} f = \int_{I^n} f, \]

Therefore, we have to define \( \int f \) for all continuous functions on \( I^n \).

Let \( 1 \leq i \leq m \). Then

\[ x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_m) \]

is a continuous function on

\([a_i, b_i] \), so we can define the Riemann integral

\[ \int_{a_i}^{b_i} f(x_1, \ldots, x_i, \ldots, x_m) \, dx_i \]

which is a function of \( x_1, x_i-1, x_i+1, \ldots, x_m \).
We define

\[ f_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) = \sum a_i \int f(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \, dx_i \]

Claim. \( f_i, 1 \leq i \leq n \), are continuous functions on 
\( [a_i, b_i] \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \ldots \times [a_m, b_m] \).

Proof. We have

\[ f_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) = \]

\[ \sum a_i \int (f(x_1, ..., x_i, ..., x_n) - f(\tilde{x}_1, ..., x_i, ..., \tilde{x}_n)) \, dx_i \]
Since $I^n$ is compact, $f$ is uniformly continuous on $I^n$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that
\[ d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) < \delta \]
\[ \Rightarrow |f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)| < \varepsilon. \]
This implies that
\[ d((x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)) < \delta \]
\[ \Rightarrow d((x_1, \ldots, x_i, \ldots, x_n), (y_1, \ldots, y_{i-1}, x_i, y_i, \ldots, y_m)) < \delta \]
\[ \Rightarrow |f_i(x_1, \ldots, x_i, \ldots, x_n) - f_i(y_1, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots, x_n)| < \varepsilon \]
\[ \Rightarrow |f_i(x_1, \ldots, x_i, \ldots, x_n) - f_i(y_1, \ldots, y_i, \ldots, y_m)| \leq \varepsilon \]
\[
\begin{align*}
\phi_i & \leq \int (f(x_1, \ldots, x_i, \ldots, x_n) - \\
& \quad \alpha_i \cdot f(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)) \, dx_i \\
& \leq \varepsilon (\theta_i - \alpha_i).
\end{align*}
\]

This proves the claim. Let \( \pi \) be a permutation of \( (1, 2, \ldots, n) \). Then, by induction we can evaluate

\[
I_{\pi}(f) = \int \left( \int_{a_{\pi(m)}}^{b_{\pi(m)}} \left( \int_{a_{\pi(m-1)}}^{b_{\pi(m-1)}} \left( \int_{a_{\pi(1)}}^{b_{\pi(1)}} f(x_1, \ldots, x_m) \, dx_{\pi(1)} \right) \, dx_{\pi(m-1)} \right) \, dx_{\pi(m-1)} \right) \, dx_{\pi(n)}. 
\]

Clearly, \( I_{\pi} : f \mapsto I_{\pi}(f) \) is a linear form on \( C(I) \).
Let \( f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) \) where \( f_i : [a_i, b_i] \to \mathbb{R} \) are continuous functions. Then \( f \) is continuous on \( \mathbb{R}^n \).

Moreover, we have

\[
\int_{a_i}^{b_i} f(x_1, \ldots, x_n) \, dx_i = \int_{a_i}^{b_i} \left( \prod_{i=1}^{n} f_i(x_i) \right) \, dx_i = \int_{a_i}^{b_i} f_i(x_i) \prod_{i=1}^{n} f_{i+1}(x_{i+1}) \, dx_i.
\]

This implies that

\[
\int_{a_i}^{b_i} f(x_1, \ldots, x_n) \, dx_i = \int_{a_i}^{b_i} f(x_i) \prod_{i=1}^{n} f_{i+1}(x_{i+1}) \, dx_i.
\]
Hence, for any two permutations \( \pi, \pi' \) we have

\[
\mathcal{I}_\pi(f) = \mathcal{I}_{\pi'}(f)
\]

for all special functions \( f \).

Let \( A \) be the subalgebra of \( C(\Omega) \) spanned by functions

\[
f(x_1, \ldots, x_m) = f_1(x_1) \cdots f_m(x_m),
\]

where \( f_i \in C([a_i, b_i]) \), \( 1 \leq i \leq m \).

Then \( f \) contains constants.

Moreover, if \( (x_1, \ldots, x_m) \) and \( (y_1, \ldots, y_m) \) are two different points in \( \Omega^m \), there exists \( 1 \leq i \leq m \), such that \( x_i \neq y_i \).
Let \( \varphi : [a_i, b_i] \rightarrow \mathbb{R} \)
\[
\varphi(x_i) \neq \varphi(y_i).
\]
Then \( f(x_1, ..., x_m) = \varphi(x_i) \) is in \( A \) and
\[
f(x_1, ..., x_m) \neq f(y_1, ..., y_m).
\]
Hence, \( A \) differs points in \( A \).
By Stone-Weierstrass theorem, \( A \) is dense in \( C(I^m) \),
Put
\[
\omega(I^m) = |b_1 - a_1| \cdots |b_m - a_m|.
\]
Then we have
\[ |I_{\pi}(f)| \leq \sum_{\pi(1)}^{(f)} \sum_{\pi(n)} f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \, dx_{\pi(1)} \cdots dx_{\pi(n)} \leq \|f\| \cdot \text{vol}(I^m) . \]

Let \( f \in C(I^m) \), and \( \varepsilon > 0 \). Then there exists \( g \in \mathcal{A} \) such that \( \|f - g\| < \varepsilon \). This implies that
\[
|I_{\pi}(f) - I_{\pi}(g)| = |I_{\pi}(f - g)| \leq \|f - g\| \cdot \text{vol}(I^m) < \varepsilon \cdot \text{vol}(I^m)
\]
for any permutation \( \pi \).

Let \( \pi \) and \( \pi' \) be two permutations. Then
\[ |I_\pi(f) - I_{\pi'}(g)| \leq \varepsilon \cdot \text{vol}(I^n) \]

and

\[ |I_{\pi'}(f) - I_{\pi'}(g)| \leq \varepsilon \cdot \text{vol}(I^n). \]

Since \( g \in A \)

\[ I_{\pi'}(g) = I_{\pi'}(g). \]

Hence, we have

\[ |I_\pi(f) - I_{\pi'}(f)| = \]

\[ |I_\pi(f) - I_{\pi'}(g) + I_{\pi'}(g) - I_{\pi'}(f)| \]

\[ \leq |I_\pi(f) - I_{\pi'}(g)| + |I_{\pi'}(f) - I_{\pi'}(g)| \]

\[ \leq 2 \cdot \varepsilon \cdot \text{vol}(I^n). \]

It follows that

\[ I_\pi(f) = I_{\pi'}(f) \]

for any \( f \in C(I^n). \)
Therefore, linear forms $I_{\pi}$ are all equal on $C(I^n)$. We define $I(f) = I_{\pi}(f)$ for any permutation $\pi$. Hence $I$ is an iterated integral and it doesn't depend on order of integration!