Inverse function theorem

Let $U \subset \mathbb{R}^m$ be an open set and $V \subset \mathbb{R}^m$ an open set.

Let $F: U \rightarrow V$ be a differentiable function and $G: V \rightarrow U$ its differentiable inverse.

Let $x_0 \in U$ and $F(x_0) = y_0$.

Then $G(y_0) = x_0$, i.e.,

$G(F(x_0)) = x_0$.

Since $G \circ F = 1$ (identify map) by chain rule.
we see that
\[ G'(y_0) \circ F'(x_0) = I, \]

By switching roles of \( F \) and \( G \) we also conclude that
\[ F'(x_0) \circ G'(y_0) = I. \]

Hence, \( F'(x_0) \) is invertible and its inverse is \( G'(y_0) \).

By fundamental theorem of linear algebra
\[ \dim \ker F'(x_0) + \dim \text{im} F'(x_0) = n. \]

Since \( F'(x_0) \) is injective
\[ \dim \ker F'(x_0) = 0, \]

and since \( F'(x_0) \) is onto
\[ \dim \text{im} F'(x_0) = m. \]

\[ \Rightarrow \quad n = m. \]
Hence, our situation is possible only if $n = m$. Moreover, we have

\[
G(y_0) = F(x_0)^{-1}
\]

The inverse function theorem is essentially the opposite statement. Let $O \subset \mathbb{R}^m$ be an open set and $F: O \rightarrow \mathbb{R}^m$ a continuously differentiable function. Let $a \in O$ be such that $F'(a)$ is invertible.
Then there exist open sets \( a \in U \subset U \) and \( V \subset \mathbb{R}^n \) such that

(i) \( F \) is a bijection of \( U \) onto \( V \);

(ii) the inverse function \( G: V \rightarrow U \) is continuously differentiable on \( V \).

If \( b = F(a) \),

\[
G'(b) = F'(a)^{-1}.
\]

Remark: Essentially the invertibility of the differential implies that locally the function has inverse.
We shall prove the theorem in a number of steps.

1. There is a neighborhood of a on which \( F \) is injective.

Let \( A = F'(a) \). Then \( A \) is invertible. Let \( y \in \mathbb{R}^m \). Consider the function \( \varphi(x) = x + A^{-1}(y - F(x)) \) on \( O \).

Then \( \varphi(x) = x \iff A^{-1}(y - F(x)) = 0 \iff y = F(x) \).
So, $y$ has a fixed point $x$ if and only if $F(x) = y$.

\[ \psi(x) = I - A^{-1}F'(x) = A^{-1}(A - F'(x)). \]

Hence \[ \| \psi'(x) \| = \| A^{-1}(A - F'(x)) \| \leq \| A^{-1} \| \cdot \| A - F'(x) \| \]

Since $F$ is continuously differentiable and $A = F'(a)$, there exists an open ball $U$ centered at $a$ such that \[ \| (A - F'(x)) \| \leq \frac{1}{2\| A^{-1} \|} \] for $x \in U$.

\[ \Rightarrow \| \psi'(x) \| \leq \frac{1}{2} \] for $x \in U$. 
since $U$ is convex, this implies that

$$|\varphi(c) - \varphi(d)| \leq \frac{1}{2} |c - d|$$

for $c,d \in U$ (by a lemma we proved in Lecture 1). Therefore, $\varphi$ is a contractor on $U$. By the contraction principle, $\varphi$ has at most one fixed point.

Hence, if $F(x) = y$ for some $x \in U$, this $x$ is unique. This implies that $F|_U$ is injective. This completes the proof of (ii).