$\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ ← "natural topology

↑

product topology

$$\|(x_1, \dots, x_n, x_{n+1})\| =$$

$$\|(x_1, \dots, x_n, 0) + (0, \dots, 0, x_{n+1})\| \le$$

$$\le \|(x_1, \dots, x_n, 0)\| + \|(0, \dots, 0, x_{n+1})\| =$$

$$= \|(x_1, \dots, x_n)\| + |x_{n+1}| .$$

$$d((x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1})) =$$

$$= d((x_1, \dots, x_n), (y_1, \dots, y_n)) + |x_{n+1} - y_{n+1}|$$

$\Rightarrow$ ball of radius ε centered at $(x_1, \dots, x_{n+1})$ contains the ball of radius $\varepsilon/2$ centered at $(x_1, \dots, x_n) \times (x_{n+1} - \varepsilon/2, x_{n+1} + \varepsilon/2)$

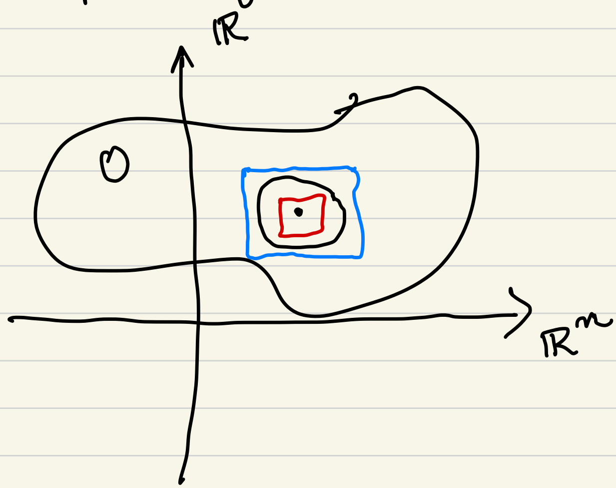
$\Rightarrow$ Open set in natural topology is open in the product topology.

$$\|(x_1, \dots, x_n, 0)\| = \sqrt{x_1^2 + \dots + x_n^2} \leq$$
$$\leq \sqrt{x_1^2 + \dots + x_{n+1}^2} = \|(x_1, \dots, x_{n+1})\|$$
$$\|(0, \dots, x_{n+1})\| = \sqrt{x_{n+1}^2} \leq$$
$$\leq \|(x_1, \dots, x_{n+1})\|$$

$\Rightarrow$

ball of radius ε centered at $(x_1, \dots, x_{n+1})$ is contained in the product of the ball of radius ε cetered at $(x_1, \dots, x_n)$ and $(x_{n+1} - \varepsilon, x_{n+1} + \varepsilon)$.

This implies that an open set in product topology is open in the natural topology. Hence, the natural topology of $\mathbb{R}^{n+1}$ is the product topology of $\mathbb{R}^n \times \mathbb{R}$



__Theorem__ $[a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$ is compact.

__Proof__: We proved this for $n=1$.

General case follows by induction. $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ as topological spaces. By induction assumption, $[a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]$ is compact. Hence,

$$([a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]) \times [a_n, b_n]$$

is compact, by the theorem we proved last time. $\blacksquare$

A set S in $\mathbb{R}^n$ is <u>bounded</u> if it is contained in some ball centered at 0 (this depends on metric, not on topology!).

__Theorem__ (Heine-Borel)

Let $S \subset \mathbb{R}^n$. The following properties are equivalent:

(i) S is compact;

(ii) S is closed and bounded.

__Proof__: $\mathbb{R}^n$ is hausdorff.

Hence, if it is compact, it is closed.

$$S \subset \bigcup_{n=1}^{\infty} B(0,n).$$

By compactness, $S \subset B(0,N)$ for some $N > 0$, i.e., S is bounded.

If S is closed and bounded, S is a closed subset of a sufficiently large box $[a_1, b_1] \times \cdots \times [a_n, b_n]$. Since this box is compact, S is compact. $\blacksquare$

Weierstrass theorem

Can define

$$\alpha : [0,1] \underset{\alpha^{-1}}{\overset{}{\rightleftarrows}} [a,b]$$

$$\alpha(t) = a + (b-a)t$$

$$\alpha^{-1}(s) = \frac{s-a}{b-a}$$ - inverse map.

Theorem: Let f be a continuous function on $[a,b]$. Then there exist function on $[a,b]$. Then for any $\varepsilon > 0$ there exists a polynomial P such that $\|f - P\| < \varepsilon$ in $\mathcal{C}([a,b])$.

(Therefore, polynomials are dense in $\mathcal{C}([a,b])$.

Proof: Can assume that $a=0$, $b=1$.

Can assume that $f(0)=f(1)=0$.

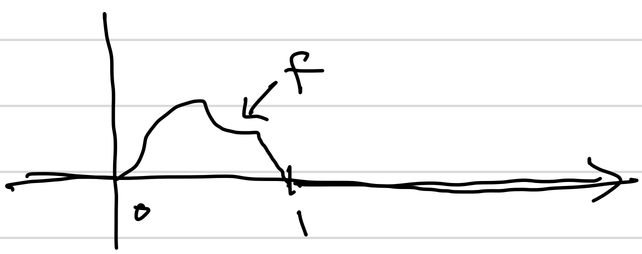
<u>Proof</u>: Put $g(x)=f(0)+(f(1)-f(0))x$ linear.

$F(x)=f(x)-g(x) \qquad F(0)=F(1)=0$.

$F(0)=f(0)-g(0)=0, \; F(1)=f(1)-g(1)=0$.

Can extend f to be 0 outside $[0,1]$.

Then f is continuous on $\mathbb{R}$.



$Q_n(x)=c_n(1-x^2)^n$

$(1-x^2)^n$ is positive on $[-1,1]$. Can pick $c_n>0$ such that

$$\int_{-1}^{1} Q_n(x)\,dx = 1.$$

$$P_n(x)=\int_{-1}^{1} f(x+t)\,Q_n(t)\,dt$$

for $0\le x\le 1$.

f is zero outside $[0,1]$ $\Rightarrow$

$t \longmapsto f(x+t)$ is 0 for $t \leq -x$, $t \geq 1-x$,

Hence

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt.$$

Make change of variables $s = t+x$

$ds = dt$

$$P_n(x) = \int_0^1 f(s)\, Q_n(s-x)\, ds,$$

$$Q_n(s-x) = c_n\left(1-(s-x)^2\right)^n$$

is polynomial in s and x

$$Q_n(s-x) = \sum a_{pq}\, s^p x^q$$

$$\Rightarrow P_n(x) = \sum a_{pq} \left(\int_0^1 f(s)\, s^p\, ds\right) x^q$$

is a polynomial in x.

$$|P_n(x) - f(x)| = \left|\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - \right.$$

$$\left. - f(x) \int_{-1}^{1} Q_n(t)\, dt\right| =$$

$$= \left| \int_{-1}^{1} \big(f(x+t) - f(x)\big) Q_n(t)\, dt \right| \leq$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt,$$

Since f is uniformly continuous on $\mathbb{R}$, there exists $\delta > 0$ such that $|t| < \delta$ implies $|f(x+t) - f(x)| < \varepsilon/2$.

$$|P_n(x) - f(x)| \leq \int_{-1}^{-\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$+ \underbrace{\int_{-\delta}^{\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt}_{< \varepsilon/2} +$$

$$+ \int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$\leq \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\, dt \leq \frac{\varepsilon}{2} \int_{-1}^{1} Q_n(t)\, dt = \frac{\varepsilon}{2}.$$

f is bounded, $\exists M > 0$ such that

$$|P_n(x)-f(x)| \le M\int_{-1}^{-\delta} Q_n(t)\,dt + \varepsilon/2 + M\int_{\delta}^{1} Q_n(t)\,dt \le \varepsilon/2 + 2M\int_{\delta}^{1} Q_n(t)\,dt$$

(since Q_n is even).

If we prove that $\int_{\delta}^{1} Q_n(t)\,dt \to 0$ (*) as $n \to \infty$, there exists n such that $2M \cdot \int_{\delta}^{1} Q_n(t)\,dt < \varepsilon/2$ for $n \geq n_0$.

Hence, for $n \ge n_0$, we have $|P_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$.

This implies that $\|P_n - f\| < \varepsilon$ for $n \geq n_0$.

It remains to prove (*)