Products

X, Y – sets

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

is the <u>product</u> of sets X and Y.

X, Y topological spaces

$\mathcal{U}, \mathcal{V}$ topologies on X, resp. Y.

We define the topology on $X \times Y$ by putting

$O \subset X \times Y$ is open if for any $(x,y) \in O$ there exist open sets $U \subset \mathcal{U}$ and $V \subset \mathcal{V}$ such that

$$(x,y) \in U \times V \subset O.$$

<u>Exercise</u>: All such sets O define a topology on $X \times Y$ - <u>product topology</u>.

$X \times Y$ equipped with product topology is called <u>product of topological spaces</u> X and Y.

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{q} & Y \\
{\scriptstyle p}\downarrow & & \\
X & &
\end{array}
\qquad
\begin{array}{l}
p(x,y) = x \\
q(x,y) = y
\end{array}
$$

- <u>projections</u>

<u>Exercise</u>: p and q are continuous maps.

X, Y

$x \in X \quad i_x(y) = (x, y)$

① $i_x : Y \longrightarrow X \times Y$ is a continuous map

② $j_y : X \longrightarrow X \times Y$

$j_y : x \longmapsto (x, y)$

is a continuous map.

<u>Theorem</u>: X, Y topological spaces, $C \subset X$, $K \subset Y$ compact sets. Then $C \times K \subset X \times Y$ is compact.

<u>Proof</u>. Let $\mathcal{U}$ be an open cover of $C \times K$.

$x \in C \quad i_x : Y \longrightarrow X \times Y$

$i_x(y) = (x, y)$ is a continuous map

$i_x(K)$ is compact.

$\quad \shortparallel$

$\{x\} \times K \subset C \times K$

$\mathcal{U}$ is a cover of $\{x\} \times K$.

Since $\{x\} \times K$ is compact there exists a finite subcover $\mathcal{U}'$ which covers $\{x\} \times K$.

$\mathcal{U}' = \{O_1, O_2, \dots, O_n\}$

For $(x, y) \in O_m$ there exist $U_x \in \mathcal{U}$, $V_y \in \mathcal{V}$ (topologies of X and Y respectively) such that $U_y \ni x$, $V_y \ni y$

$U_y \times V_y \subset O_m$.

This defines a cover of

$\{x\} \times K$.

Since $\{x\} \times K$ is compact

we can find a subcover

$(U_p \times V_q \,;\ 1 \leq p \leq N,\ 1 \leq q \leq M)$

of $\{x\} \times K$

Put $U = U_1 \cap \dots \cap U_N \ni x$

is a neighborhood of x

$y \in K$, $(x,y) \in U_p \times V_q$

$\Longrightarrow (x,y) \in U \times V_q$

$(U \times V_q \,;\ 1 \leq q \leq M)$ is a cover of $\{x\} \times K$

$U \times V_q \subset U_p \times V_q \subset O_m$ for some O_m.

Therefore, $\{O_1, \dots, O_n\}$ is also a cover of $U \times K$!

We can do this construction for any $x \in C$.

For each x, we get a finite subcover of $\mathcal{U}$ of $U^x \times K$.

$(U^x; x \in C)$ are an open cover of C. Take a finite subcover $(U^{x_s}; 1 \le s \le L)$

$(U^{x_p} \times K)$ covers $C \times K$

each is covered by a finite subcover of $\mathcal{U}$.

$\Longrightarrow$ $C \times K$ is covered by a finite subcover of $\mathcal{U}$. $\blacksquare$