K compact space

$\mathcal{C}(X)$ – all continuous functions $f : X \longrightarrow \mathbb{R}$.

$f, g \in \mathcal{C}(X)$

$$(f+g)(x) = f(x) + g(x), \quad x \in X$$

$f+g \in \mathcal{C}(X)$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$f \cdot g \in \mathcal{C}(X)$

To prove this we first remark the following fact

$x \longmapsto |f(x)|$ is also continuous (since $|.|$ is a continuous function. Let $x \in X$. Take $M > 0$ such that $|f(x)| < M$

then $(-\infty, M]$ is a neighborhood of $|f(x)|$. Hence, there exists a neighborhood $U \ni x$ such that $y \in U \Longrightarrow |f(y)| \leq M$.

Now,

$$|(f \cdot g)(x) - (f \cdot g)(y)| =$$

$$|f(x) \cdot g(x) - f(y) \cdot g(y)| =$$

$$= |(f(x) - f(y)) \cdot g(x) + (g(x) - g(y)) \cdot f(y)|$$

$$\leq |f(x) - f(y)| \cdot |g(x)| + |g(x) - g(y)| \cdot |f(y)|$$

$$\leq |f(x) - f(y)| \cdot |g(x)| + |g(x) - g(y)| \cdot M$$

$$\leq |f(x) - f(y)| \cdot N + |g(x) - g(y)| \cdot M$$

where $N > |g(x)|$.

By continuity of f and g

there exists a neighborhood U of x such that

$$|f(x)-f(y)| < \frac{\varepsilon}{2N}$$

$$|g(x)-g(y)| < \frac{\varepsilon}{2M}$$

for $y \in U$.

$$\Rightarrow \quad |(f\cdot g)(x) - (f\cdot g)(y)| < \varepsilon$$

for $y \in U$.

Hence, $f \cdot g$ is continuous on X.

$\mathcal{C}(X)$ is closed under addition and multiplication. Since constant functions are continuous on X, it is also closed under multiplication by real numbers.

Norm satisfies

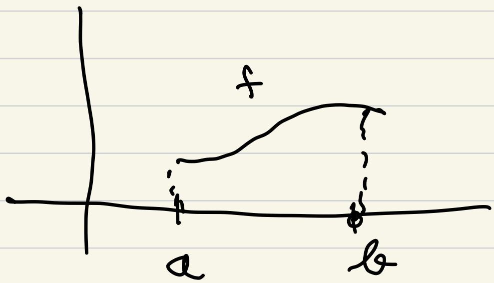
1. $\|f\| \geq 0 \quad \|f\| = 0 \Leftrightarrow f = 0.$
2. $\|\underset{\in \mathbb{R}}{\alpha} \cdot f\| = |\alpha| \cdot \|f\|$
3. $\|f+g\| \leq \|f\| \cdot \|g\|$
4. $\|f \cdot g\| \leq \|f\| \cdot \|g\|$

$\mathcal{C}(X)$ is a <u>normed algebra</u> over $\mathbb{R}$.

Topology is given by metric

$$d(f,g) = \|f-g\|.$$

$K = [a,b]$



$\mathcal{C}([a,b])$ is a normed algebra

Denote by $\mathcal{A}$ the subset of $\mathcal{C}([a,b])$ which consists of restrictions of polynomials

$$P(x) = \sum_{n=0}^{N} a_n x^n$$

to $[a,b]$ (N is the degree of P, and it is arbitrary). Since sum and product of polynomials are polynomials, the subset

$\mathcal{A}$ is a subalgebra of $\mathcal{C}([a,b])$.

$\mathcal{C}([a,b])$ is a metric space with metric $d(f,g) = \|f-g\|$.

$\Rightarrow$ <u>topological space</u>.

<u>Def</u>. Let X be a topological space and Y a subset of X. We say that Y is <u>dense</u> in X if $\overline{Y} = X$.

<u>Example</u>: $\mathbb{R}$ with natural topology. $\mathbb{Q}$ is dense in $\mathbb{R}$.

We are going to prove the following theorem.

Weierstrass' theorem.

The subalgebra $\mathcal{A}$ is dense in $\mathcal{C}([a,b])$.

This implies the following:
Let f be a continuous function on $[a,b]$. Let $\varepsilon > 0$. Then there exists a polynomial P such that

$$|f(x) - P(x)| < \varepsilon$$

for all $x \in [a,b]$.
f can be uniformly approximated by polynomials.