Last time we proved
K - compact subset of $\mathbb{R}$
$\Rightarrow$ K is closed and bounded.

Now we want to prove the converse.

<u>Theorem</u>. Let $a, b \in \mathbb{R}$. Then $[a,b]$ is compact subset of $\mathbb{R}$.

<u>Proof</u>. Let $\mathcal{U}$ be an open cover of $[a,b]$.

Let $x \in \mathbb{R}$, $a \leq x \leq b$. Then $\mathcal{U}$ is an open cover of $[a,x]$.

Let A be the set of all $x \in [a,b]$ such that $[a,x]$

is covered by a finite subcover of $\mathcal{U}$.

① A is not empty

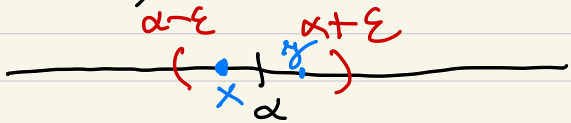
$[a,a]=\{a\}$ is in some $U\in\mathcal{U}$.

Therefore $\{U\}$ is a finite subcover.

② A is bounded above (by b), hence $\exists\ \alpha=\sup A$.

$\boxed{\alpha\in A}$

Let U be an open set in $\mathcal{U}$ which contains α. Then there exists $\varepsilon>0$ such that $(\alpha-\varepsilon,\alpha+\varepsilon)\subset U$.



If $A \cap (\alpha-\varepsilon, \alpha+\varepsilon) = \emptyset$
α cannot be the least upper bound of A.
Hence, $A \cap (\alpha-\varepsilon, \alpha+\varepsilon) \neq \emptyset$.
Take $x \in A \cap (\alpha-\varepsilon, \alpha+\varepsilon)$.
Then there exists a finite subcover $\mathcal{U}'$ of $\mathcal{U}$ which covers $[a, x]$. Moreover

$$[x, \alpha] \subset (\alpha-\varepsilon, \alpha+\varepsilon) \subset U.$$

Hence $\mathcal{U}' \cup \{U\}$ is a finite subcover of $[a, \alpha] = [a, x] \cup [x, \alpha]$.
It follows that

$$\alpha \in A.$$

③ $\alpha = b$.

Assume that $\alpha < b$. Then there exists $y \in \mathbb{R}$, $\alpha \leq y < \alpha + \varepsilon \leq b$ for some small ε such that $(\alpha - \varepsilon, \alpha + \varepsilon) \subset U \in \mathcal{U}$.

Let $\mathcal{U}''$ be a finite subcover of $\mathcal{U}$ covering $[a, \alpha]$. Then $\mathcal{U}'' \cup \{U\}$ is a finite subcover covering $[a, y] = [a, \alpha] \cup [\alpha, y]$. Hence, $y \in A$. This contradicts $\alpha = \sup A$.

It follows that $\alpha = b$.

Therefore, there exists a finite subcovering of $\mathcal{U}$ covering $[a,b]$. It follows that $[a,b]$ is compact. ∎

<u>Corollary</u>: Let $K \subset \mathbb{R}$. Then the following statements are equivalent:

(i) K is compact;

(ii) K is closed and bounded.

<u>Proof</u>. We proved (i) $\Rightarrow$ (ii).
If (ii) holds, K is a closed subset of $[a,b]$. By the theorem, $[a,b]$ is compact.
$\Rightarrow$ K is compact. ∎

Let $f \in \mathcal{C}(X)$. Consider

$$x \longmapsto f(x) \xmapsto{|\cdot|} |f(x)|$$

This is a composition of two continuous functions $\Rightarrow$ continuous on X.

$$\|f\| = \max_{x \in X} |f(x)|$$

① $\|f\| \geq 0$

② $\|f\| = 0 \Longrightarrow |f(x)| = 0$ for all x
$\Rightarrow f(x) = 0$ for all $x \in X$, $f = 0$!

③ $\alpha \in \mathbb{R}$

$$\|\alpha f\| = \max_{x \in X} |(\alpha \cdot f)(x)| =$$

$$= \max_{x \in X} \left[|\alpha| \cdot |f(x)| \right] = |\alpha| \cdot \max_{x \in X} |f(x)| =$$

$$= |\alpha| \cdot \|f\|.$$

(4)

$$\|f+g\| = \max_{x\in X} |f(x)+g(x)| \le$$

$$\le \max_{x\in X} \left(|f(x)| + |g(x)| \right) \le$$

$$\le \max_{x\in X} |f(x)| + \max_{x\in X} |g(x)| =$$

$$\|f\| + \|g\|$$

This is a <u>norm</u> of $\mathcal{C}(X)$.

$$d(f,g) = \|f-g\|$$

is a <u>metric</u> on $\mathcal{C}(X)$.

Defines a topology on $\mathcal{C}(X)$

– <u>topology of uniform convergence</u>.