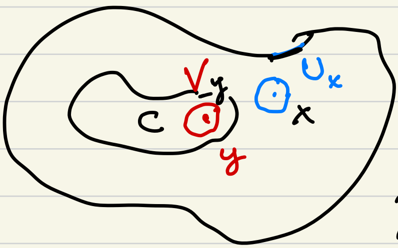
Lemma. If X is hausdorff any compact subset C of X is closed.

Proof. Let $x \notin C$. Let $y \in C$. Then $x \neq y$. Since X is hausdorff,



there exist open sets $U_x \ni x$ and $V_y \ni y$ such that $U_x \cap V_y = \emptyset$. $(V_y \,;\, y \in C)$ is an open cover of C. Since C is compact, There exists a finite subcover $(V_{y_i} \,;\, 1 \leq i \leq n)$.

$U = U_1 \cap \dots \cap U_n$ is an open set containing x.

$\Rightarrow$ neighborhood of x,

$$V_i \cap U \subseteq V_i \cap U_i = \emptyset$$

$$\Rightarrow \left(\bigcup V_i\right) \cap U = \emptyset$$

$$\Rightarrow C \cap U = \emptyset$$

$$\Rightarrow U \subset X \setminus C.$$

This implies that $X \setminus C$ is open. Hence C is closed. $\blacksquare$

<u>Theorem</u>: Let $f: X \to Y$ be a continuous map. Let C be a compact set in X. Then $f(C)$ is compact.

<u>Proof</u>: Let $(V_i ; i \in I)$ be an open cover of $f(C)$.

Then $f^{-1}(V_i)$, $i \in I$, are open sets in X since f is continuous.

Moreover, since $f(C) \subset \bigcup_{i \in I} V_i$ we have

$$C \subset f^{-1}\Big(\bigcup_{i \in I} V_i\Big) = \bigcup_{i \in I} f^{-1}(V_i)$$

i.e. $(f^{-1}(V_i) ; i \in I)$ is an open cover of C. Since C is compact, the cover has a finite subcover, i.e.,

$$C \subset f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}).$$

$$\Rightarrow \quad f(C) \subset U_{i_1} \cup \dots \cup U_{i_n}.$$

Hence, $(U_{i_1}, U_{i_2}, \dots, U_{i_n})$ is a finite subcover of $(U_i ; i \in I)$. ∎

<u>Lemma</u>. Let S be a compact subset of $\mathbb{R}$. Then S is closed and bounded.

<u>Proof</u>. Since $\mathbb{R}$ is hausdorff, S is closed.

Let $I_n = (-n, n)$, $n \in \mathbb{N}$.

Then $\mathcal{Y} = (I_n; n \in \mathbb{N})$ is an open cover of $\mathbb{R}$. Hence, it is an open cover of S.

Since S is compact, $\mathcal{Y}$ has a finite subcover.

This implies that

$$S \subset I_n = (-n,n) \subseteq [-n,n]$$

for some $n \in \mathbb{N}$. Hence, S is bounded. $\blacksquare$

Let X be a compact topological space. Denote by $\mathcal{C}(X)$ the set of all continuous functions $f : X \longrightarrow \mathbb{R}$.

$\mathcal{C}(X)$ is a vector space.

$f, g \in \mathcal{C}(X)$

$x \in X$, f and g are continuous at x

Let $\varepsilon > 0$. Then there exist neighborhoods U and V of x such that $y \in U \Longrightarrow |f(y) - f(x)| < \varepsilon/2$

$y \in V \Longrightarrow |g(y) - g(x)| < \varepsilon/2$.

Hence for $y \in U \cap V$ (neighborhood of x)

$|(f+g)(y) - (f+g)(x)| =$

$|f(y) + g(y) - f(x) - g(x)| \leq$

$\leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The function $f+g$ is continuous at x. Hence $f+g \in \mathcal{C}(X)$.

$(f,g) \longrightarrow f+g$ is an additive operation on $\mathcal{C}(X)$.

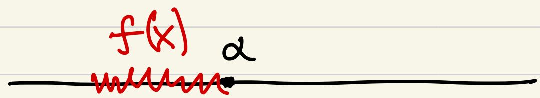
<u>Exercise</u>. $\alpha \in \mathbb{R}$ $\quad (\alpha \cdot f)(x) = \alpha \cdot f(x)$
$x \in X$. Show that if $f \in \mathcal{C}(X)$ then $\alpha \cdot f \in \mathcal{C}(X)$.

This defines multiplication of functions by real numbers.

<u>Exercise</u>: Check that $\mathcal{C}(X)$ is a vector space.

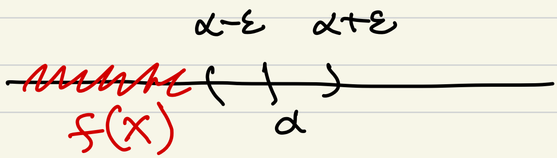
Let $f: X \longrightarrow \mathbb{R}$ be in $\mathscr{C}(X)$.
Then $f(X)$ is compact.
Therefore, $f(X)$ is a bounded and closed subset in $\mathbb{R}$.
Since it is bounded

$\alpha = \sup f(X)$



exists. Assume that $\alpha \notin f(X)$.
Since $f(X)$ is closed. Hence, $\mathbb{R} \setminus f(X)$ is open. Since $\alpha \in \mathbb{R} \setminus f(X)$, there exists $\varepsilon > 0$ such that
$(\alpha-\varepsilon, \alpha+\varepsilon) \subset \mathbb{R} \setminus f(X)$; i.e.

$$(\alpha-\varepsilon, \alpha+\varepsilon) \cap f(X) = \emptyset$$



This

contradicts that α is the least upper bound of $f(X)$.

Hence, $\alpha \in f(X)$

$$\alpha = \max_{x \in X} f(x),$$

<u>Exercise</u>: Prove that

$$\beta = \inf f(X)$$

satisfies

$$\beta = \min_{x \in X} f(x),$$

Therefore,

$$\min_{x \in X} f(x) \le f(x) \le \max_{x \in X} f(x),$$

Continuous functions on compact spaces attain their maxima and minima.