Continuous maps

\((X, U), (Y, V)\) topological spaces

\(f : X \rightarrow Y\)

A map \(f\) is continuous at \(x_0 \in X\) if for any neighborhood \(N\) of \(f(x_0)\), \(f^{-1}(N)\) is a neighborhood of \(x_0\).

\(f : X \rightarrow Y\) is a continuous map if it is continuous at all \(x \in X\).
Claim. $f : X \to Y$ is continuous if and only if for any open set $V \subseteq Y$, $f^{-1}(V)$ is open.

Proof. Assume that $f$ is continuous. Let $V$ be an open set in $Y$. If $V \cap f(x) = \emptyset$, $f^{-1}(V) = \emptyset$ is open.

If $V \cap f(x)$ is not empty, $f^{-1}(V) = \{ x \in X \mid f(x) \in V \}$ is not empty. If $x \in f^{-1}(V)$, $f(x)$ is in $V$. $V$ is an open neighborhood of $f(x)$ $\Rightarrow$ $f$ is continuous at $x$ $\Rightarrow$ $f^{-1}(V)$ is a neighborhood of $x$.
\( f^{-1}(V) \) is a neighborhood of any of its points \( \Rightarrow f^{-1}(V) \) is open.

Assume that \( f \) satisfies \( V \) open in \( Y \) \( \Rightarrow f^{-1}(V) \) is open in \( X \) for all \( V \subseteq Y \).

Let \( x \in X \). Let \( N \) be a neighborhood of \( f(x) \). Then there exists open set \( V \) such that \( f(x) \in V \subseteq N \)

\[ \Rightarrow x \in f^{-1}(V) \subseteq f^{-1}(N) \]

open

\( \Rightarrow f^{-1}(N) \) is a neighborhood of \( x \).
$f$ is continuous at $x$.

Category of topological spaces

Objects - topological spaces
Morphisms - continuous maps

- Identity map is continuous
- Composition of continuous maps is continuous

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\parallel & & \parallel \\
\downarrow{g} & & \downarrow{e} \\
\downarrow{g \circ f} & & \downarrow{z}
\end{array}
\]
Exercises:

1. If $(X, d_X)$ and $(Y, d_Y)$ are two metric spaces $f : X \rightarrow Y$ is continuous at $x_0$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x_0, x) < \delta \implies d(f(x_0), f(x)) < \varepsilon.$$

2. If $Y$ has chaos topology any maps $f : X \rightarrow Y$ is continuous.

3. $(X, U)$ topological space $Y \subset X$ subset $V = \{ Y \cap U \mid U \in U \}$ is a topology on $Y$, induced topology on $Y$. 
compact sets
X topological space
Y ⊆ X subset. U family of open sets in X. U is an open cover of Y if
Y ⊆ ∪ U.
U ∈ U
A set C is compact if any open cover of C has a finite subcover.
(i.e. if U is an open cover there exists a finite subset of U such that F is a cover).
Lemma. Let $C$ be a compact subset of $X$ and $Z \subseteq C$ a closed set in $X$. Then $Z$ is also compact.

Proof: Let $U$ be an open cover of $Z$. Then $X \setminus Z$ is open. $U' = U \cup \{X \setminus Z\}$ is a cover of $C$. Since $C$ is compact, $U'$ has a finite subcover $F$ of $C$.

If $F$ is a subset of $U$ it is a subcover of $U$ which covers $C \Rightarrow$ covers $Z$. If $F$ contains
\[ S = (S \cup U) \cup \{x \in Z \} \]

\[ C = U \cup U \cup (x \in Z) \quad \forall \in \mathcal{F} \cup U \]

\[ \Rightarrow Z \subset U \cup U \quad \forall \in \mathcal{F} \cup U \]

\[ \Rightarrow \mathcal{F} \cup U \text{ is a finite open cover of } Z. \text{ Hence } Z \text{ is compact.} \]

**Example.** \( X \) equipped with chaos topology - Any subset of \( X \) is compact.