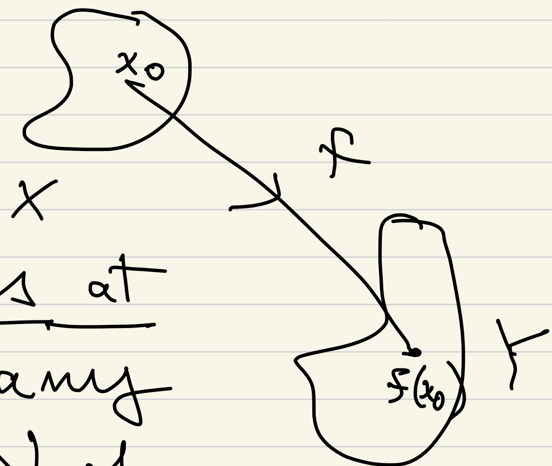
Continuous maps

$(X, \mathcal{U})$, $(Y, \mathcal{V})$
topological spaces

$f : X \longrightarrow Y$
a map.



f is <u>continuous at</u>
$x_0 \in X$ if for any
neighborhood N of
$f(x_0)$, $f^{-1}(N)$ is a neighborhood
of x_0.

$f : X \longrightarrow Y$ is a <u>continuous</u>
<u>map</u> if it is continuous at
all $x \in X$.

<u>Claim</u>. $f: X \longrightarrow Y$ is continuous if and only if for any open set $V \subset Y$, $f^{-1}(V)$ is open.

<u>Proof</u>. Assume that f is continuous. Let V be an open set in Y. If $V \cap f(X) = \emptyset$, $f^{-1}(V) = \emptyset$ is open.

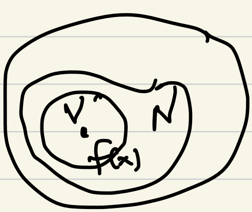
If $V \cap f(X)$ is not empty, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is not empty. If $x \in f^{-1}(V)$, $f(x)$ is in V. V is an open neighborhood of $f(x) \Longrightarrow f$ is continuous at $x \Longrightarrow f^{-1}(V)$ is a neighborhood of x

$f^{-1}(V)$ is a neighborhood of any of its points $\Longrightarrow$ $f^{-1}(V)$ is open.

Assume that f satisfies V open in $Y \Rightarrow f^{-1}(V)$ is open in X for all $V \in \mathcal{V}$.

Let $x \in X$. Let N be a neighborhood of $f(x)$. Then there exists open set V such that

$$f(x) \in V \subset N$$



$$\Rightarrow x \in \underbrace{f^{-1}(V)}_{\text{open}} \subset f^{-1}(N)$$

$\Longrightarrow f^{-1}(N)$ is a neighborhood of x.

f is continuous at x.

$\blacksquare$

Category of topological spaces

objects – topological spaces

morphisms – continuous maps

- identity map is continuous
- composition of continuous maps is continuous

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z
\end{tikzcd}
$$

Exercises:

① If (X, d_X) and (Y, d_Y) are two metric spaces $f: X \longrightarrow Y$ is continuous at x_0 if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x_0, x) < \delta \Longrightarrow d(f(x_0), f(x)) < \varepsilon.$$

② If Y has chaos topology any map $f: X \longrightarrow Y$ is continuous.

③ $(X, \mathcal{U})$ topological space $Y \subset X$ subset

$$\mathcal{V} = \{ U \cap Y \mid U \in \mathcal{U} \}$$

is a topology on Y.

- induced topology on Y

Compact sets

X topological space

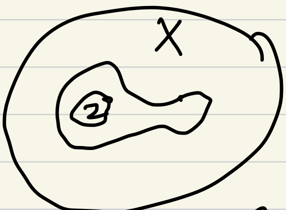
$Y \subset X$ subset. $\mathcal{U}$ family of open sets in X. $\mathcal{U}$ is an <u>open cover</u> of Y if

$$Y \subset \bigcup_{U \in \mathcal{U}} U.$$

A set C is <u>compact</u> if any open cover of C has a <u>finite</u> subcover. (i.e. if $\mathcal{U}$ is an open cover there exists a finite subset of $\mathcal{U}$ such that $\mathcal{F}$ is a cover).

<u>Lemma</u>. Let C be a compact subset of X and $Z \subset C$ a closed set in X. Then Z is also compact.

<u>Proof</u>: Let $\mathcal{U}$ be an open cover of Z. Then $X \setminus Z$ is open $\mathcal{U}' = \mathcal{U} \cup \{X \setminus Z\}$ is a cover of C. Since C is



compact, $\mathcal{U}'$ has a finite subcover $\mathcal{F}$ of C.

If $\mathcal{F}$ is a subset of $\mathcal{U}$ it is a subcover of $\mathcal{U}$ which covers $C \Rightarrow$ covers Z. If $\mathcal{F}$ contains

$$\mathcal{F}' = (\mathcal{F} \cap \mathcal{U}) \cup \{X \setminus Z\}$$

$$C \subset \bigcup_{U \in \mathcal{F} \cap \mathcal{U}'} U \cup (X \setminus Z)$$

$$\Rightarrow Z \subset \bigcup_{U \in \mathcal{F} \cap \mathcal{U}} U$$

$\Rightarrow$ $\mathcal{F} \cap \mathcal{U}$ is a finite open cover of Z. Hence Z is compact. $\blacksquare$

<u>Example</u>. X equipped with chaos topology. Any subset of X is compact.