Lie groups

G - Lie group

$m : G \times G \longrightarrow G$ multiplication

G has a distinguished point 1

- the identity

$T_1(G)$ - tangent space at 1.

Sophus Lie discovered that

$T_1(G)$ has additional structure

which reflects the multiplication

in G.

Tangent space of the product

X, Y two manifolds

$$\begin{array}{cc} X, & Y \\ \Psi & \Psi \\ x & y \end{array} \qquad (x,y) \in X \times Y$$

$$T_{(x,y)}(X \times Y) = ?$$

$X \xrightarrow{i} X \times Y \qquad i(x) = (x, y), \ x \in X$

$Y \xrightarrow{j} X \times Y \qquad j(y) = (x, y), \ y \in Y$

$p : X \times Y \to X, \quad p(x, y) = x$

$q : X \times Y \to Y, \quad q(x, y) = y$

$p \circ i = id_X \qquad q \circ j = id_Y$

$\underbrace{(p \circ j)(y) = x \qquad (q \circ i)(x) = y}_{\text{constant maps}}$

$\Rightarrow T_{(x,y)}(p) \circ T_x(i) = I_{T_x(X)}$

$T_{(x,y)}(p) \circ T_x(j) = 0$

$T_{(x,y)}(q) \circ T_y(j) = I_{T_y(Y)}$

$T_{(x,y)}(q) \circ T_x(i) = 0$

$T_x(i) : T_x(X) \to T_{(x,y)}(X \times Y)$

is injective

Analogously

$$T_y(j) : T_y(Y) \longrightarrow T_{(x,y)}(X \times Y)$$

is injective

$$T_x(i) \oplus T_y(j) : T_x(X) \oplus T_y(Y)$$

$$\to T_{(x,y)}(X \times Y)$$

is injective

$$0 = \left(T_x(i) \oplus T_y(j)\right)(\xi, \eta) =$$

$$= T_x(i)\xi + T_x(j)\eta$$

Apply $T_{(x,y)}(\phi) \Longrightarrow 0 = \xi$

Apply $T_{(x,y)}(g) \Longrightarrow 0 = \eta$.

$$\dim\left(T_x(X) \oplus T_y(Y)\right) =$$

$$\dim T_x(X) + \dim T_x(Y) =$$

$$= \dim_x X + \dim_x Y = \dim_{(x,y)}(X \times Y) =$$

$$= \dim T_{(x,y)}(X \times Y).$$

$$\Rightarrow \quad T_x(X) + T_y(Y) = T_{(x,y)}(X \times Y).$$

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1 $\quad G \times G \xrightarrow{m} G$

$$T_1(G) \oplus T_1(G) \xrightarrow{T_{(1,1)}(m)} T_1(G)$$

$$\begin{tikzcd} i_1 : G \arrow[r] \arrow[rr, bend right, "id_G"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$$(\xi, 0) \longrightarrow \xi$$

$$(0, \eta) \longrightarrow \eta$$

$$(\xi, \eta) = (\xi, 0) + (0, \eta) \longrightarrow \xi + \eta$$

$$\boxed{T_{(1,1)}(m)(\xi, \eta) = \xi + \eta}$$

Doesn't say anything about m!

② $g \in G \quad \varphi_g(h) = g \cdot h g^{-1}$

$$T_1(\varphi_g) : T_1(G) \longrightarrow T_1(G)$$

differentiate

$$T_1(G) \times T_1(G) \longrightarrow T_1(G).$$

$$G = GL(n, \mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

open submanifold of

Tangent vectors at I

$$t \longmapsto I + tS$$

$$\varphi_A(I + tS) = A(I + tS)A^{-1} =$$

$$= I + tASA^{-1}$$

$$T_1(\varphi_A)(S) = ASA^{-1}.$$

$$A = I + sT \qquad A^{-1} = I - sT + \frac{s^2}{2}T^2 - \dots$$

$$T_1(\varphi_A)S = (I + sT)\, S\, (I - sT + \dots) =$$

$$= I + s(TS - ST) + s^2 \dots$$

$\Rightarrow$ tangent vector

$$TS - ST = [T, S]$$

<u>Lie bracket</u>.

$M_n(\mathbb{R})$ with $[\, , \,]$

is a <u>Lie algebra</u>

- bilinear operation
- $[T, S] = -[S, T]$ <u>anticommutative</u>
- not associative

$$[T, [S, P]] + [S, [P, T]] + [P, [T, S]] = 0$$

<u>Jacobi identity</u>.