Hence $f \longmapsto T(f \circ F)$ is a tangent vector of $b$.

It follows that we defined a map $T_a(F) : T_a(M) \longrightarrow T_b(N)$

$$\boxed{T_a(F)(T)(f) = T(f \circ F)}$$

for $T \in T_a(M)$.

The map $T_a(F) : T_a(M) \longrightarrow T_b(N)$ is called the *derivative* or *differential* of $F$ at $a$.

The map $T_a(F)$ is linear.

$T_a(F)(\alpha T + \beta S)(f) =$

$= (\alpha T + \beta S)(f \circ F) = \alpha T(f \circ F) +$

$$\beta S(f\circ F) = \left(\alpha T_a(F)T + \beta T_a(F)S\right)(f)$$

2

If $id : M \longrightarrow M$ , $id(a) = a$

$$T_a(id)(T)(f) = T(f\circ id) = T(f)$$

$$\boxed{T_a(id) = I_{T_a(M)}.}$$

Chain rule :

$$\underset{a}{M} \xrightarrow[F]{} \underset{b}{N} \xrightarrow[G]{} \underset{c}{Q}$$

$$T_a(G\circ F)(T)(f) =$$

$$T(f\circ G\circ F) = T_a(F)(T)(f\circ G) =$$

$$= \left(T_b(G)\,T_a(F)\right)(T)(f)$$

$$\Longrightarrow \quad \boxed{T_a(G\circ F) = T_b(G)\circ T_a(F)}$$

If $F : M \longrightarrow N$ is a diffeomorphism

$$F \circ F^{-1} = id_N$$

$$F^{-1} \circ F = id_M$$

$$\Longrightarrow \quad T_a(F) \circ T_b(F^{-1}) = I_{T_b(N)}$$

$$T_b(F^{-1}) \circ T_a(F) = I_{T_a(M)}$$

$$\Longrightarrow \quad \boxed{T_b(F^{-1}) = T_a(F)^{-1}}$$

In particular, $T_a(F)$ is an isomorphism and

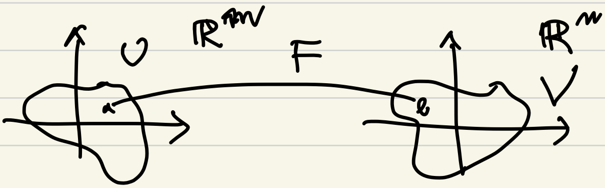
$$\dim T_a(M) = \dim T_b(N).$$

Let $M$ be a manifold, $a \in M$

$c = (U, \varphi, n)$ a chart around $a$.

Then $\varphi : U \longrightarrow \varphi(U)$ is a diffeomorphism

$$\Rightarrow \dim T_a(M) = n = \dim_a M.$$




$$T_a(F)(\partial_i)(f) = \partial_i(f \circ F) = \sum_{j=1}^{n} (\partial_j f)(F(a)) \cdot$$

$$\frac{\partial F_j}{\partial x_i}(a) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_i}(a) \cdot \underbrace{\partial_j}_{\text{basis of } T_b(N)}(f)$$

The matrix of $T_a(F)$ is

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}.$$