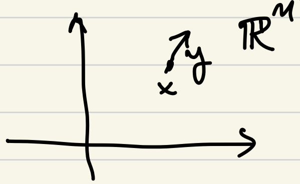
Tangent spaces



directional derivative

$$\frac{d}{dt} f(x+ty)\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\cdot y_i$$

A <u>tangent vector</u> at a is

a linear map $T: C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$

such that it satisfies

$$T(fg) = T(f)\,g(a) + f(a)\,T(g)$$

- <u>Leibniz rule</u>. All tangent vectors

at a form a vector space over $\mathbb{R}$.

T, S – tangent vectors

$$(T+S)(fg) = T(fg) + S(fg) =$$

$$T(f)\,g(a) + f(a)\,T(g) + S(f)\,g(a) + f(a)\,S(g) =$$

$$= (T(f)+S(f))\,g(a) + f(a)\,(T(g)+S(g)) =$$

$= (T+S)(f)\,g(a) + f(a)\,(T+S)(g)$

$T+S$ is a tangent vector

$\lambda \in \mathbb{R}$, λT is a tangent vector.

$T_a(\mathbb{R}^n)$ – the vector space of all tangent vectors at a – the <u>tangent space</u> at a.

$$e_i : f \longmapsto \left(\frac{\partial}{\partial x_i} f\right)(a) = (\partial_i f)(a)$$

is a tangent vector at a

① $(e_1, \dots, e_n)$ <u>are linearly independent</u>

$$\sum_{i=1}^{n} \lambda_i e_i = 0$$

$$0 = \left(\sum_{i=1}^{n} \lambda_i e_i\right)(x_j) = \lambda_j (\partial_j x_j)(a) = \lambda_j.$$

$\lambda_j = 0$ for all $1 \le j \le n$.

② From Taylor's formula we can rove that for any function $f \in C^\infty(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$ we have

$$f(x) = f(a) + \sum_{i=1}^{n} (x_i - a_i) g_i(x)$$

This implies

$$(\partial_j f)(a) = 0 + g_j(a)$$

Let $T \in T_a(\mathbb{R}^n)$. Then

$$T(1) = T(1 \cdot 1) = 1 \cdot T(1) + T(1) \cdot 1 = 2T(1)$$

$$\Rightarrow T(1) = 0.$$

$$T(f) = \underbrace{T(f(a) \cdot 1)}_{=0} + T\left(\sum_{i=1}^{n} (x_i - a_i) \cdot g_i\right) =$$

$$= \sum_{i=1}^{n} \Big(T(x_i - a_i) \cdot g_i(a) + 0 \cdot T(g_i) \Big) =$$

$$= \sum_{i=1}^{n} T(x_i) \cdot (\partial_j f)(a) = \sum_{i=1}^{n} T(x_i) \cdot e_i(f)$$

Hence, $T = \sum_{i=1}^{n} c_i e_i$, $c_i \in \mathbb{R}$, and $(e_1, \dots, e_n)$ is a basis of $T_a(\mathbb{R}^n)$. Therefore,

$$\dim T_a(\mathbb{R}^n) = n.$$

M is a differentiable manifold $a \in M$. A linear form $T : C^\infty(M) \to \mathbb{R}$ is a tangent vector at a if

$$T(f \cdot g) = f(a) T(g) + T(f) g(a).$$

As before, all tangent vectors at a form a vector space $T_a(M)$ – the <u>tangent space</u> of M at a.

Let N be another manifold and $F: M \longrightarrow N$ a differentiable map. Let $b = F(a)$.

Consider $T \in T_a(M)$. Let $f \in C^\infty(N)$. Then $f \circ F \in C^\infty(M)$. Hence we can define

$$f \longmapsto T(f \circ F)$$

as a linear form on $C^\infty(N)$.

Moreover

$$T((f \cdot g) \circ F) = T((f \circ F) \cdot (g \circ F)) =$$

$$= T(f \circ F) \cdot (g \circ F)(a) + (f \circ F)(a)\, T(g \circ F) =$$

$$= T(f \circ F)\, g(\underset{\overset{\shortparallel}{b}}{F(a)}) + f(\underset{\overset{\shortparallel}{b}}{F(a)})\, T(g \circ F)$$

Hence $f \longmapsto T(f \circ F)$ is a tangent vector of b.

It follows that we defined a map $T_a(F) : T_a(M) \longrightarrow T_b(N)$

$$\boxed{T_a(F)(T)(f) = T(f \circ F)}$$

for $T \in T_a(M)$.

The map $T_a(F) : T_a(M) \longrightarrow T_b(N)$ is called the <u>derivative</u> or <u>differential</u> of F at a.