Dimension of a manifold

Connectedness - Let X be a topological space. X is **connected** if X is not equal to $U \cup V$ where U, V are nonempty open, disjoint sets.

Example: Open ball in $\mathbb{R}^n$ is connected.

Lemma. Let $x \in X$. Denote by $\mathcal{U}$ the family of all connected subsets of X containing x. Then the union of all sets in $\mathcal{U}$ is connected.

Proof. Assume that Y is that

union, $Y = U \cap V$, $U, V \neq \emptyset$
and $U \cap V = \emptyset$, U, V open in Y.
Let $Z \in \mathcal{U}$. Then $Z \cap U$, $Z \cap V$
are open in Z, $(Z \cap U) \cap (Z \cap V) = \emptyset$
and $(Z \cap U) \cup (Z \cap V) = Z$.
Since Z is connected, this is
possible only if $Z \cap U = \emptyset$ or
$Z \cap V = \emptyset$.

Assume that $x \in U$. Then $x \in Z \cap U$
$\Rightarrow Z \cap V = \emptyset \Rightarrow Z \subset U$.
Hence, union of all Z is in U.
$\Rightarrow V = \emptyset$. Therefore, Y is
connected. $\blacksquare$

The set Y is the <u>connected component</u> of x.

Let $y \in Y$. Then the connected component W of y contains Y (since it is connected).

It follows that $y \in Y \subset W$.

Therefore, $W \in \mathcal{U}$ and $W \subset Y$.

Hence, $W = Y$.

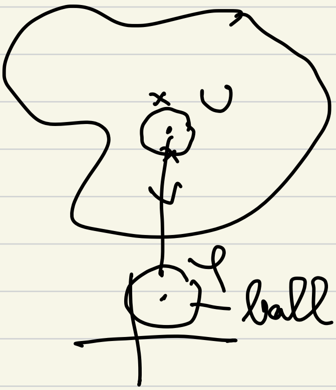
$\Rightarrow$ Y is connected component of each of its points.

X is a disjoint union of all of its connected components.

<u>Theorem</u>. Connected components of a differentiable manifold

are open (and closed).

<u>Proof</u>. Let M be a manifold and N a connected component of M. Let $x \in N$. Then there



exists a chart $c = (U, \varphi, x)$ around x such that $\varphi(U)$ is a ball. $\Longrightarrow$ $\varphi(U)$ is connected $\Rightarrow$ U is connected. $\Rightarrow$ $U \subset N$. $\Rightarrow$ N is open. All connected components are open. $X \setminus N$ is a union of components. $\Longrightarrow$ N is also closed. $\boxtimes$

M manifold, $x \in M$

$c = (U, \varphi, n)$ chart around x.

If $d = (V, \psi, m)$ is another chart around x

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a diffeomorphism $\Longrightarrow \boxed{n = m}$

$\Rightarrow \dim_x M = n$ – <u>dimension</u> of M at x.

(i) $x \longmapsto \dim_x M$ is a locally constant function (it is constant on a neighborhood of x).

Assume that $\dim_x M$ has two different values n, m.

Then $U = \{x \in M \mid \dim_x M = n\}$

$V = \{x \in M \mid \dim_x M \neq n\}$

are open sets in M, $U \cup V = M$

$U, V \neq \phi$ and $U \cap V = \phi$.

Hence M is not connected.

$\Longrightarrow$ Local dimension $\dim_x M$ is constant on connected components of a manifold.

If the manifold is connected, $\dim_x M = \dim M$ dimension of M.

Products.

M, N are manifolds

$M \times N$ - product

- topological space
- $U \subset M \times N$ is open if for any $(x,y) \in U$, there exist open $U_x \ni x$ in M, $V_y \ni y$ in N such that $U_x \times V_y \subset U$.
- define charts on $M \times N$

$c = (U, \varphi, m)$ $d = (V, \psi, n)$

$c \times d = (U \times V, \varphi \times \psi, m+n)$

This defines on $M \times N$ a structure of differentiable manifold. - <u>product manifold</u>

of M and N,

Lie groups

A Lie group G is a

(a) group;

(b) manifold;

$m : M \times M \longrightarrow M$

multiplication — product manifold

is differentiable map

$i : M \longrightarrow M \quad i(a) = a^{-1}$

is a differentiable map.