Last time we considered the map $S : \mathcal{O} \longrightarrow \mathbb{R}^n$ which is continuously differentiable and $S(0) = 0$; moreover

$$S = (x_1, \dots, x_{m-1}, \beta_m(x), \dots, \beta_n(x))$$

and $\partial_m \beta_m(0) \neq 0$.

This implies that $\beta_i(0) = 0$ for $1 \leq i \leq m$.

Define a map

$$G(x_1, \dots, x_n) = (x_1, \dots, x_{m-1}, \beta_m(x), \dots x_n)$$

Then $G(0) = 0$, G is continuously differentiable and

$$G'(x) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & \mathcal{O} & \\ & & 1 & & & \\ & & & \partial_m \beta_m(x) & & \\ & \mathcal{O} & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

Hence

$$J_G(x) = \partial_\mu \beta_\mu(x)$$

and $J_G(0) \neq 0$. Hence by the inverse function theorem, G is a bijection of an open neighborhood of 0 onto another and the inverse function is continuously differentiable.

Finally, G is primitive.

Clearly

$$(S \circ G^{-1})(G(x)) = S(x)$$

for x in some neighborhood of 0, i.e.,

$$(S\circ G^{-1})(x_1,\dots,\beta_m(x),\dots,x_n)$$

$$= \left(x_1,\dots,x_{m-1},\beta_m(x),\beta_{m+1}(x),\dots,\beta_n(x)\right)$$

If we change variables

$$(x_1,\dots,\beta_m(x),\dots,x_n) = (y_1,\dots,y_n)$$

we get

$$(S\circ G^{-1})(y_1,\dots,y_n) =$$

$$=(y_1,\dots,y_m,\gamma_{m+1}(y),\dots,\gamma_n(y))$$

Hence

$$T_1 = S\circ G^{-1}$$

is the function of the same type as T but with m replaced by $m+1$.

This implies that
$$T_1 \circ G = S.$$
Hence
$$T = F \circ S = F \circ T_1 \circ G.$$
<u>Claim</u>. The map T is a composition of flips and primitive maps (on some neighborhood of 0 in $\mathbb{R}^n$).

<u>Proof of the claim</u>.
We prove this by downward induction in m. We already remarked that

for $m=n$ the map T is primitive.

Assume that for $m+1$ the map T is a composition of flips and primitive maps. Then, by the formula we just established the map T for m is also a composition of primitive maps and flips.

This proves our claim.

Consider an arbitrary map $T: \mathcal{O} \longrightarrow \mathbb{R}^n$ which is continuously differentiable, a bijection and $T'(x)$ is invertible for all $x \in \mathcal{O}$.

Let $a \in \mathcal{O}$ and $b = T(a)$

Then, the map

$$S(x) = T(x+a) - b$$

is a continuously differentiable map on $\mathcal{O} - a$ and

$$S'(x) = T'(x+a)$$

by the chain rule.

Since $S(0) = 0$, by the preceding discussion in some neighborhood of 0, S is a composition of flips and primitive maps.

This implies that in some neighborhood of a the map T is a composition of translations, flips and primitive maps.

Since the change of variables formula holds for translations, flips and primitive maps, the change of variables formula holds for functions supported in some neighborhood of b. Using partition of unity argument it follows that the formula holds in general.