① <u>Translations</u>

If $T(x_1, \dots, x_n) = (x_1 + c_1, \dots, x_n + c_n)$ for some $c \in \mathbb{R}^n$, we say that T is a <u>translation</u>

Clearly

$$T'(x) = I$$

$$J_T(x) = 1$$

and the change of variables formula follows from 1-dimensional case.

② <u>Flips</u>

Let i, j be two indices, $1 \le i < j \le n$. A <u>flip</u> is a map

$$T(x_1, \dots, x_i, \dots, x_j, \dots, x_n) =$$
$$= (x_1, \dots, x_j, \dots, x_i, \dots, x_n)$$

Clearly,

$$
T'(x) = \begin{array}{c}
\begin{array}{ccccccccc} & & & i & & j & & & \end{array} \\
\left[\begin{array}{ccccccccc}
1 & & & 0 & & 0 & & & \\
 & \ddots & & \vdots & 0 & \vdots & & 0 & \\
0 & \cdots & & 0 & \cdots & 1 & \cdots & & \\
 & 0 & & \vdots & 1 & \vdots & & 0 & \\
 & & & & & \ddots & & & \\
 & \cdots & & 1 & \cdots & 0 & \cdots & 0 & \\
 & & & & & & 1 & & \\
 & 0 & & \vdots & 0 & \vdots & 0 & \ddots & \\
 & & & & & & & & 1
\end{array}\right]
\end{array}
\begin{array}{c} \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\ \end{array}
$$

i.e., it is the identity matrix with i^{th} and j^{th} column switched. Therefore

$$J_T(x) = \det T'(x) =$$

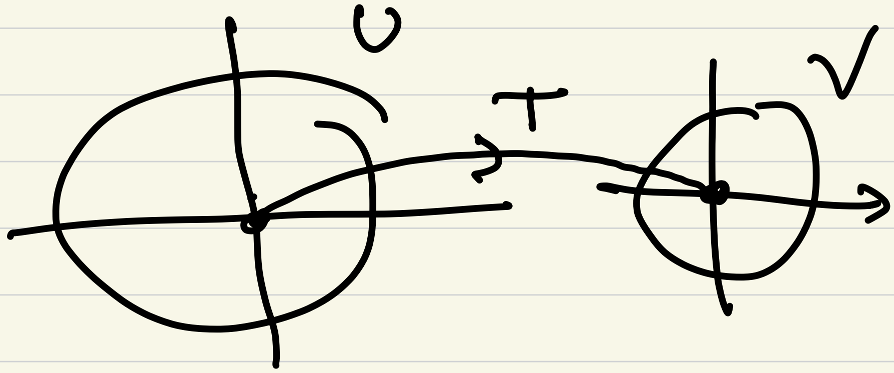
$$= -1.$$

Hence $|J_T(x)| = 1$

and the change of variables formula follows from the independence of the integral on the order of integration.

Now we want to describe arbitrary map T <u>locally</u> as a composition of "simpler" maps. First, composing it with translations, we can assume that $0 \in U$ and $0 \in V$ and

$$T(0) = 0.$$



Assume that

$$T(x_1, \ldots, x_n) =$$

$$(x_1, \ldots, x_{m-1}, \alpha_m(x), \ldots, \alpha_n(x))$$

with $\alpha_i : U \to \mathbb{R}$ continuously differentiable. Then, if $m=1$, T is arbitrary; and if $m=n$, T is a primitive map. We shall now discuss intermediate cases.

First, since $T(0)=0$,

$\alpha_i(0)=0$ for $m \leq i \leq n$.

Moreover, we have

$$T'(x) = \left[\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \partial_1\alpha_m(x) & \cdots & \cdots & \cdots & \cdots & \partial_n\alpha_m(x) \\ \vdots & & & & & \\ \partial_1\alpha_n(x) & \cdots & \cdots & \cdots & \cdots & \partial_m\alpha_n(x) \end{array}\right]$$

hence

$$J_T(x) = \left|\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & & & \\ & \ddots & & & & 0 & \\ & & \ddots & & & & \\ 0 & \cdots & \cdots & 1 & & & \\ \hline \partial_1\alpha_m(x) & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \cdots & & \cdots & & \cdots & & \partial_m\alpha_m(x) \end{array}\right| =$$

$$= \begin{vmatrix} \partial_m \alpha_m(x) & \cdots & \partial_n \alpha_m(x) \\ \vdots & & \vdots \\ \partial_m \alpha_n(x) & \cdots & \partial_n \alpha_n(x) \end{vmatrix} \neq 0$$

not all coefficients in this column can vanish at 0!

Therefore, there exists k, $m \leq k \leq n$, such that $(\partial_m \alpha_k)(0) \neq 0$!

Let F be the flip which switches x_m and x_k. Then $S = F \circ T$ is a continuously differentiable bijection such that

$$S(x_1,\dots,x_n) = F(T(x_1,\dots,x_n)) =$$

$$= (x_1,\dots,x_{m-1}, \underset{\beta_m(x)}{\underset{\shortparallel}{\alpha_k(x)}}, \dots, \alpha_m(x), \dots \underset{\beta_n(x)}{\alpha_n(x)})$$

i.e.

$$S(x_1,\dots,x_n) = (x_1,\dots,x_{m-1}, \beta_m(x), \dots, \beta_n(x))$$

and $\partial_m \beta_m(0) = \partial_m \alpha_k(0) \neq 0$.

Moreover,

$$T = F \circ S.$$

Therefore, T is a composition of a flip and a map S which has the additional property $(\partial_m \beta_m)(0) \neq 0$.