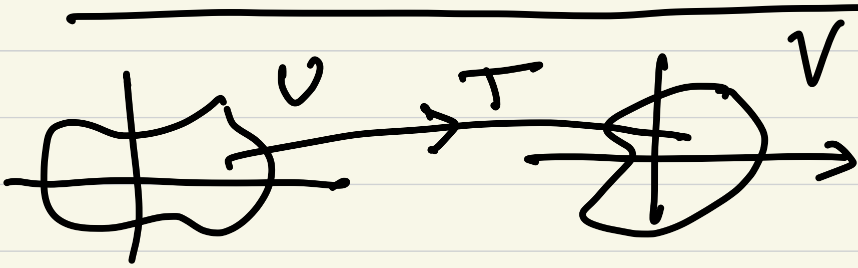
Second reduction



Assume that for any $a \in V$ there exists an open neighborhood V_a of a, $V_a \subset V$ such that the formula holds for functions $f \in C_0(\mathbb{R}^n)$ such that

$\operatorname{supp} f \subset V_a$.

Let $f \in C_0(\mathbb{R}^n)$ with

$K = \operatorname{supp} f \subset V$.

Then $(V_a; a \in K)$ is an open cover of K. Let $(\varphi_1, \dots, \varphi_m)$ be a partition of unity subordinated to the cover $(V_a; a \in K)$.

Then

$$\sum_{i=1}^{m} \varphi_i(y) = 1 \text{ for } y \in K.$$

$\Rightarrow$

$$f(y) = \sum_{i=1}^{m} \varphi_i(y) f(y) =$$

$$= \sum_{i=1}^{m} (\varphi_i f)(y), \quad y \in \mathbb{R}^n$$

Put $f_i = f \cdot \varphi_i$.

Then $\operatorname{supp} f_i \subset V_{a_i}$ for some $a_i \in K$.

By the assumption, we have

$$\int_{\mathbb{R}^n} f_i(T(x)) \, |J_T(x)| \, dx =$$

$$= \int f_i(x) \, dx$$

for $1 \leq i \leq m$. Hence

$$\int_{\mathbb{R}^n} f(x) \, dx = \int \sum_{i=1}^{m} f_i(x) \, dx =$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^n} f_i(x) \, dx =$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^n} f_i(T(x)) \, |J_T(x)| \, dx =$$

$$= \int_{\mathbb{R}^n} \sum_{i=1}^{m} f_i(T(x))\, |J_T(x)|\, dx =$$

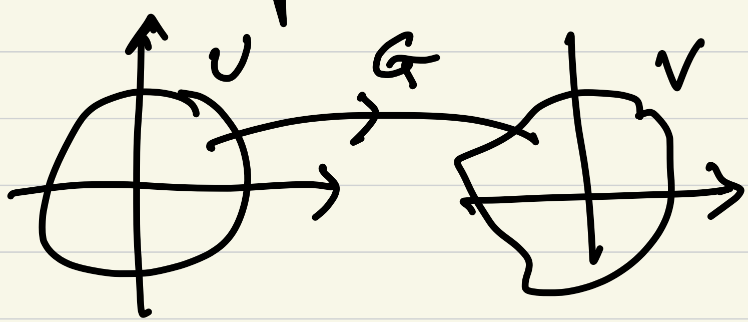
$$= \int_{\mathbb{R}^n} f(T(x))\, |J_T(x)|\, dx$$

So, the formula holds $\triangleleft$ for f.

(This is a typical example of reducing a global statement to local using partition of unity.)

Primitive maps

Let $G : U \to V$, U and V open in $\mathbb{R}^n$. Then



$G(x) = (G_1(x), \dots, G_n(x))$ where

$G_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $1 \le i \le n$.

We say that G is primitive if there exists m, $1 \le m \le n$, such that $G_i(x_1, \dots, x_n) = x_i$ for $1 \le i \le n$, $i \neq m$.

In this case $G_m(x) = g(x)$ is a function on U.

Assume, in addition, that

G is a continuously differentiable bijection of U onto V. Then

$$G'(x) = \begin{bmatrix} 1 & & & \vdots & & & \\ & 1 & 0 & \vdots & & 0 & \\ 0 & & \ddots & \vdots & & & \\ & & & 1 & & & \\ \partial_1 g(x) & \cdots & \cdots & \partial_m g(x) & \cdots & \cdots & \partial_n g(x) \\ & & & \vdots & 1 & & 0 \\ & 0 & & \vdots & & \ddots & \\ & & & \vdots & 0 & & 1 \end{bmatrix}$$

Hence, we have

$$J_G(x) = \det G'(x) = \partial_m g(x).$$

Assume that $J_G(x) \neq 0$ on U. Then we have

$$\partial_m g(x) \neq 0$$

on U,

Let f be a continuous

function with compact support in V. Then $f \circ G$ is a continuous function with compact support in U. Moreover, we have

$$\int f(G(x))\, |J_G(x)|\, dx =$$

$$\int\int_{\dots}\left(\int f(G(x))\, |\partial_m g(x)|\, dx_m\right) dx \dots dx$$

since the result doesn't depend on order of integration. Fix $x_1, \dots, x_{m-1}, x_{m+1}, \dots x_n$. Then $\gamma(x_m) = g(x_1, \dots, x_m, \dots, x_n)$ is a differentiable function on an open set in $\mathbb{R}$

$$\gamma'(x_m) = (\partial_m g)(x_1, \dots, x_m, \dots, x_n)$$

Put
$$\varphi(x_m) = f(x_1,\dots,x_m,\dots x_n).$$
Then
$$\varphi(\gamma(x_m)) = f(x_1,\dots,g(x),\dots x_n)$$
Hence, by 1-dim. version of change of variables formula, we know that

$$\int f(x_1,\dots,x_{m-1},g(x_1,\dots,x_m,\dots,x_n),\dots,x_n)$$
$$|J_G(x_1,\dots,x_{m-1},x_m,\dots x_n)|\,dx_m =$$

$$= \int \varphi(\gamma(x_m))\,|\gamma'(x_m)|\,dx_m =$$

$$= \int \varphi(x_m)\,dx_m =$$

$$= \int f(x_1,\dots,x_m,\dots,x_n)\,dx_m$$

Plugging this in the previous expression we get

$$\int f(G(x))\,|J_G(x)|\,dx =$$

$$= \int f(x)\,dx.$$

Hence, the change of variables formula holds for primitive maps.