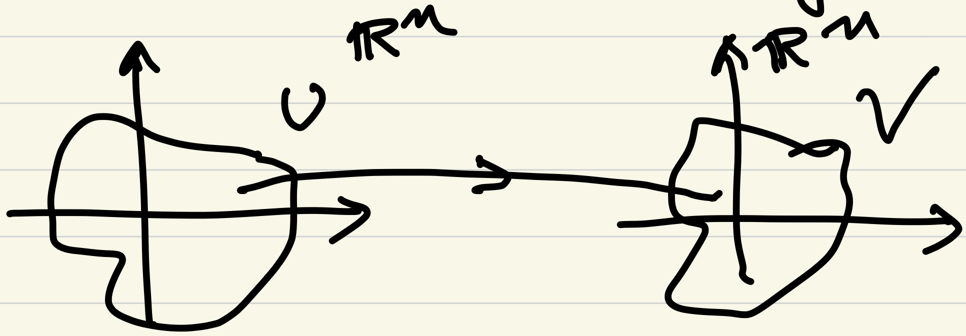
Change of variables

Let U, V open sets in $\mathbb{R}^n$

$T : U \longrightarrow V$ continuously differentiable bijection



Then $T'(x)$ is a continuous function on U

$$J_T(x) = \det T'(x)$$

is a continuous function on U.

Assume that $J_T(x) \neq 0$ for all $x \in U$.

then $T'(x)$ is invertible for any $x \in U$. Therefore, by inverse function theorem, the inverse function $T^{-1}: V \to U$ is continuously differentiable.

Moreover,

$$J_{T^{-1}}(T(x)) =$$

$$\det (T^{-1})'(T(x)) = \det (T'(x))^{-1} =$$

$$= \frac{1}{\det(T'(x))} = J_T(x).$$

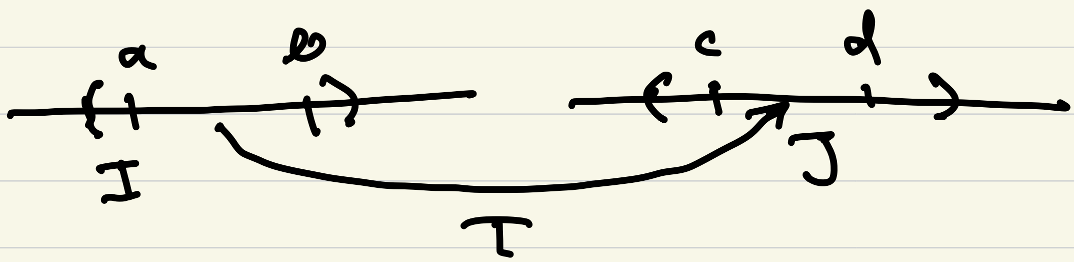
Since T^{-1} is continuous, it maps compact sets into compact sets.

Let $f \in C_0(\mathbb{R}^n)$, supp $f \subset V$.
Then $f \circ T$ is a continuous function on U.
If $K = \operatorname{supp} f$, $f(T(x)) \neq 0$ implies $T(x) \in K$, i.e. $x \in T^{-1}(K)$. As we remarked $T^{-1}(K)$ is compact.
Therefore, $\operatorname{supp}(f \circ T)$ is a closed subset of $T^{-1}(K)$, i.e. it is compact.
We want to prove the following formula

$$\boxed{\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} f(T(y))\,|J_T(y)|\,dy}$$

This is the <u>change of variables formula</u>.

Before proving the formula we discuss the case of $n = 1$.



I and J are open intervals, T is continuously differentiable.

Since $T'(x) \neq 0$ for all $x \in I$ we have two cases.

Either (a) $T'(x) > 0$ for all $x \in I$

or (b) $T'(x) < 0$ for all $x \in I$.

In the case (a) T is strictly increasing. Therefore

$T(a) = c$ and $T(b) = d$

$$\int_a^b f(T(x))\, T'(x)\, dx = \int_c^d f(x)\, dx$$

In the second case, T is strictly decreasing. Therefore

$T(a) = d$ $T(b) = C$

$$\int_a^b f(T(x))\,T'(x)\,dx =$$

$$= \int_d^c f(x)\,dx = -\int_c^d f(x)\,dx.$$

Hence,

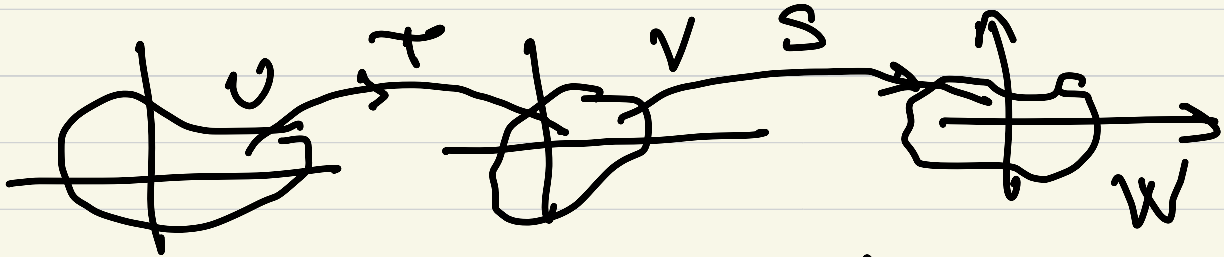
$$\int_c^d f(x)\,dx = \int_a^b f(T(x))\,|T'(x)|\,dx.$$

Hence in both cases

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(T(x))\,|T'(x)|\,dx,$$

and this is a special case of change of variables formula.

First reduction



Assume that T and S are continuously differentiable bijections of U onto V, and V onto W respectively.

Also assume that

$$J_T(x) \neq 0 \text{ for all } x \in U$$

$$J_S(y) \neq 0 \text{ for all } y \in V.$$

Then $P = S \circ T$ is a continuous bijection of U onto W. Moreover, by chain rule

$$P'(x) = S'(T(x)) \cdot T'(x)$$

for any $x \in U$. It follows that

$$J_P(x) = \det P'(x) =$$
$$= \det\big(S'(T(x)) \cdot T'(x)\big) =$$
$$= \det(S'(T(x))) \cdot \det T'(x) =$$
$$= J_S(T(x)) \cdot J_T(x)$$

for $x \in U$.

Let f be a continuous function with compact support in W. Then $f \circ S$ has compact support in V, and $f \circ P$ has compact support in U. Therefore

$$\int_{\mathbb{R}^n} f(P(x))\,|J_P(x)|\,dx =$$

$$= \int_{\mathbb{R}^n} f(S(T(x))\,|J_S(T(x))|\cdot$$

$$|J_T(x)|\,dx =$$

$$= \int_{\mathbb{R}} f(S(y))\,|J_S(y)|\,dy =$$

(using the change of variables formula for S)

$$= \int_{\mathbb{R}} f(z)\,dz$$

(using the change of variables formula for T)

Therefore if $P = S \circ T$ and the formula holds for T and S, it also holds for their composition $P = S \circ T$.