**Change of variables**

Let $U, V$ open sets in $\mathbb{R}^n$

$T: U \to V$ continuously differentiable bijection

Then $T'(x)$ is a continuous function on $U$

$$J_T(x) = \det T'(x)$$

is a continuous function on $U$.

Assume that $J_T(x) \neq 0$ for all $x \in U$. 
then $T'(x)$ is invertible for any $x \in U$. Therefore, by inverse function theorem, the inverse function $T^{-1}: V \to U$ is continuously differentiable.

Moreover,

$$\begin{align*}
J_{T^{-1}}(T(x)) &= \det(\nabla^2 T^{-1}(T(x))) = \det(\nabla^2 T'(x))^{-1} = 1 \\
&= \frac{1}{\det(\nabla^2 T'(x))} = J_T(x).
\end{align*}$$

Since $T^{-1}$ is continuous, it maps compact sets into compact sets.
Let $f \in C_0(\mathbb{R}^n)$, supp $f \subset V^3$.

Then $f \circ T$ is a continuous function on $U$.

If $K = \text{supp } f$, $f(T(x)) \neq 0$ implies $T(x) \in K$, i.e. $x \in T^{-1}(K)$. As we remarked, $T^{-1}(K)$ is compact.

Therefore, supp($f \circ T$) is a closed subset of $T^{-1}(K)$, i.e. it is compact.

We want to prove the following formula.
This is the change of variables formula.

Before proving the formula we discuss the case of $m = 1$.

$I$ and $J$ are open intervals, $T$ is continuously differentiable.
Since $T'(x) \neq 0$ for all $x \in I$ we have two cases. Either (a) $T'(x) > 0$ for all $x \in I$ or (b) $T'(x) < 0$ for all $x \in I$. In the case (a) $T$ is strictly increasing. Therefore $T(a) = c$ and $T(b) = d$

$$\int_a^b f(T(x)) T'(x) \, dx = \int_c^d f(x) \, dx$$

In the second case, $T$ is strictly decreasing. Therefore $T(a) = d$ and $T(b) = c$.
\[
\int_{a}^{b} f(T(x)) T'(x) \, dx = \int_{c}^{d} f(x) \, dx = -\int_{c}^{d} f(x) \, dx.
\]

Hence,
\[
\int_{c}^{d} f(x) \, dx = \int_{a}^{b} f(T(x)) |T'(x)| \, dx.
\]

Hence in both cases
\[
\int_{R}^{R} f(x) \, dx = \int_{R}^{R} f(T(x)) |T'(x)| \, dx,
\]

and this is a special case of change of variables formula.
First reduction

Assume that $T$ and $S$ are continuously differentiable bijections of $U$ onto $V$, and $V$ onto $W$ respectively.

Also assume that

$J_T(x) \neq 0$ for all $x \in U$

$J_S(y) \neq 0$ for all $y \in V$.

Then $P = S \circ T$ is a continuous bijection of $U$ onto $W$. Moreover,
by chain rule
\[ P'(x) = S'(T(x)) \cdot T'(x) \]
for any \( x \in U \). It follows that
\[
\begin{align*}
\mathcal{J}_P(x) &= \det P'(x) = \\
&= \det (S'(T(x))) \cdot T'(x) = \\
&= \det (S'(T(x))) \cdot \det T'(x) = \\
&= J_S(T(x)) \cdot J_T(x)
\end{align*}
\]
for \( x \in U \).

Let \( f \) be a continuous function with compact support in \( W \). Then \( f \circ S \) has compact support in \( V \), and \( f \circ P \) has compact support in \( V \).
support in $U$. Therefore
\[ \int_{\mathbb{R}^n} f(P(x)) |J_p(x)| \, dx = \]
\[ = \int_{\mathbb{R}^n} f(S(T(x)) |J_S(T(x))| \cdot |J_T(x)| \, dx = \]
\[ = \int f(x) |J_S(x)| \, dx = \]
\[ = \int f(z) \, dz \]
(\text{using the change of variables formula for $S$})
\[ \int f(z) \, dz \]
(\text{using the change of variables formula for $T$})
Therefore if \( P = S \circ T \) and the formula holds for \( T \) and \( S \), it also holds for their composition \( P = S \circ T \).