Recall of Linear Algebra

V linear space over $\mathbb{R}$

(1) <u>inner product</u> on V

$(\cdot|\cdot) : V \times V \longrightarrow \mathbb{R}$

① linear in both variables

② $(u|v) = (v|u)$ symmetry

$u, v \in V$

③ $(u|u) \geq 0$, $(u|u) = 0 \iff u = 0$.

<u>Theorem</u> (Cauchy–Schwartz inequality)

Let $u, v \in V$. Then

$$|(u|v)| \leq \|u\| \cdot \|v\|.$$

$\|u\| = (u|u)^{1/2}$

<u>norm</u> of u

Can assume that $u, v \neq 0$.

Consider the function

$$\mathbb{R} \ni t \longmapsto (tu+v \mid tu+v) \geq 0$$

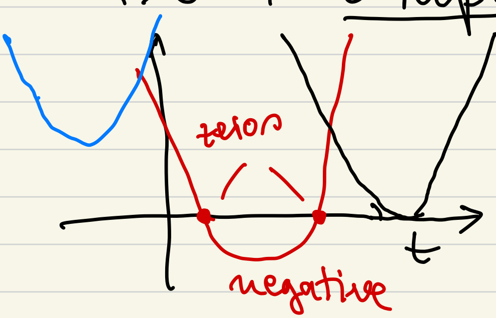
$$(tu+v \mid tu+v) = t^2 (u \mid u) + 2t(u \mid v) + (v \mid v) =$$

$$= \underset{\vee\, 0}{\|u\|^2} t^2 + 2(u \mid v)\, t + \|v\|^2$$

This is a quadratic polynomial

$$t \longmapsto at^2 + bt + c$$

$a > 0$. Graph:



Since $at^2 + bt + c \geq 0$

The red case cannot happen

Hence the quadratic equation

$$at^2 + bt + c = 0$$

has at most one real

zero $\quad x_1, x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow b^2 - 4ac \leq 0$$

$$\Rightarrow b^2 \leq 4ac$$

$$\cancel{4}(u|v)^2 \leq \cancel{4}\,\|u\|^2 \cdot \|v\|^2$$

$$|(u|v)| \leq \|u\| \cdot \|v\| \;, \quad \square$$

Let $u, v \in V$

$$\|u+v\|^2 = (u+v|u+v) =$$

$$= \|u\|^2 + 2(u|v) + \|v\|^2 \leq$$

$$\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 =$$

$$= (\|u\| + \|v\|)^2$$

Taking square root we get

$$\|u+v\| \leq \|u\| + \|v\|$$

(triangle inequality for norm).

$$d(u,v) = \|u-v\|$$

$d : V \times V \longrightarrow \mathbb{R}$

is a <u>metric</u> on V.

V is a metric space.

<u>Example</u>. $V = \mathbb{R}^n$

$$(x|y) = \sum_{i=1}^{n} x_i y_i$$

inner product in $\mathbb{R}^n$

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

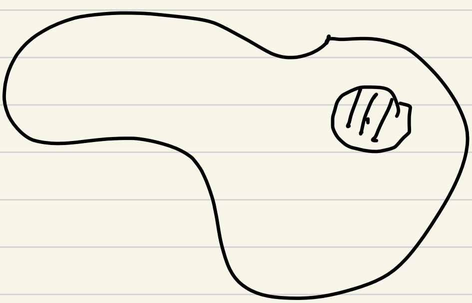
$$d(x,y) = \|x-y\| =$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

<u>Euclidean distance</u> in $\mathbb{R}^n$

It is a metric on $\mathbb{R}^n$ which defines the "natural" topology of $\mathbb{R}^n$.

Open set O in $\mathbb{R}^n$



For any $x \in O$ there exists $\varepsilon > 0$ such that

$$B(x,\varepsilon) = \{ y \in \mathbb{R}^n \mid \|x-y\| < \varepsilon \}$$

is in O.