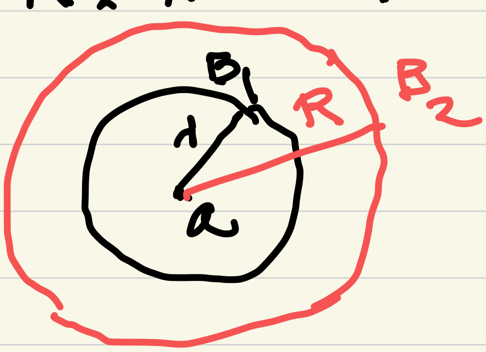
Partition of unity

Partition of unity is a construction which is used to reduce proofs of global statements to local statements.

Remark. Let $a \in \mathbb{R}^n$. Let B_1 be a an open ball of radius r centered in a B_2 - open ball of radius R centered at a

$R > r$.



Claim: There exists a continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$

such that

(a) $0 \le \varphi(x) \le 1$, $x \in \mathbb{R}^n$;

(b) $\varphi|_{B_1} = 1$;

(c) $\varphi(x) = 0$ for $x \notin B_2$.

Hence, $\operatorname{supp} \varphi \subset \overline{B_2}$.

<u>Proof</u>:

Let $\beta : \mathbb{R}_+ \to \mathbb{R}$ be the function given by this graph. Then β is continuous. The function $\varphi(x) = \beta(|x-a|)$ has properties (a), (b) and (c). ☒

Let K be a compact set in $\mathbb{R}^n$. Let $(V_\alpha ; \alpha \in A)$ be an open cover of K.

Then there exist continuous functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le m$, such that

(a) $0 \le \psi_i(x) \le 1$ for all $x \in \mathbb{R}^n$ and $1 \le i \le m$;

(b) $\operatorname{supp} \psi_i$ are compact and $\operatorname{supp} \psi_i \subset V_\alpha$ for some $\alpha \in A$;

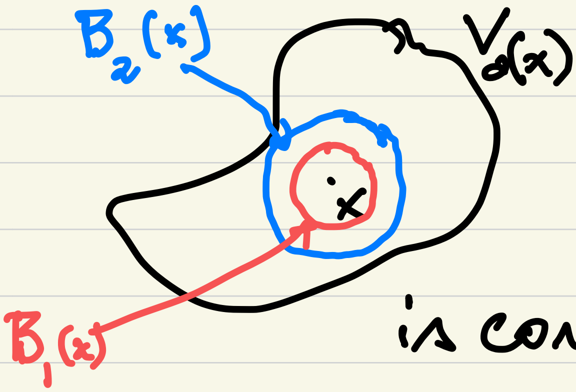
(c) $\sum_{i=1}^{m} \psi_i(x) = 1$ for all $x \in K$.

The family $(\psi_1, \dots, \psi_m)$ is called <u>partition of unity</u> on K <u>subordinate</u> to cover $(V_\alpha ; \alpha \in A)$

because of (b) because of (c)

<u>Construction</u>. $x \in K$

x is in some $V_{\alpha(x)}$, $\alpha(x) \in A$.



There exists $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous, $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^n$, $\varphi|_{B_1(x)} = 1$, $\operatorname{supp} \varphi \subset \overline{B}_2(x) \subset V_{\alpha(x)}$.

Clearly, $(B_1(x) \,;\, x \in K)$ form an open cover of K. Since K is compact, there exists a finite set $x_1, \dots, x_m \in K$ such that $(B_1(x_i) \,;\, 1 \le i \le m)$ is a subcover.

Denote by $(\varphi_1, \dots, \varphi_m)$ the

corresponding functions.
Then, $0 \leq \varphi_i(x) \leq 1$,
$\varphi_i|_{B_1(x_i)} = 1$, and
$\operatorname{supp} \varphi_i \subset \overline{B_2(x_i)} \subset V_{\alpha(x_i)}$.
Put

$$\psi_1 = \varphi_1 ;$$
$$\psi_{i+1} = (1-\varphi_1) \cdots (1-\varphi_i)\varphi_{i+1}$$
for $i \geq 2$.

By induction, this defines a family $(\psi_1, \psi_2, \ldots, \psi_m)$ of continuous functions on $\mathbb{R}^n$.

Clearly, $0 \leq \psi_1(x) \leq 1$ for all $x \in \mathbb{R}^n$.

We have $0 \leq \varphi_i(x) \leq 1$ for all $x \in \mathbb{R}^n$. Then

$(1-\varphi_i)(x) = 1 - \varphi_i(x)$

satisfies $0 \leq 1 - \varphi_i(x) \leq 1$ for all $x \in \mathbb{R}^n$. Therefore

$\Longrightarrow \psi_{i+1}(x) = (1-\varphi_1(x)) \dots (1-\varphi_i(x))\, \varphi_{i+1}(x)$

satisfies $0 \leq \psi_{i+1}(x) \leq 1$.

Hence, we have

(a) $0 \leq \psi_i(x) \leq 1$ for $x \in \mathbb{R}^n$.

Moreover, by definition

$\varphi_i(x) = 0 \Longrightarrow \psi_i(x) = 0$

hence $\operatorname{supp} \psi_i \subseteq \operatorname{supp} \varphi_i \subseteq \overline{B_2(x_i)} \subset V_{\alpha(x_i)}$

in particular $\operatorname{supp} \psi_i$ are compact. Hence (b) holds.

We claim that

$$\psi_1 + \psi_2 + \dots + \psi_i =$$
$$1 - (1-\varphi_1)(1-\varphi_2)\dots(1-\varphi_i).$$

The proof is by induction

For $i = 1$

$$\psi_1 = 1 - (1 - \varphi_1)$$

is obvious. Assume that the claim holds for i

$$\psi_1 + \psi_2 + \dots + \psi_i + \psi_{i+1} =$$
$$= 1 - (1-\varphi_1)\dots(1-\varphi_i) + \psi_{i+1} =$$
$$= 1 - (1-\varphi_1)\dots(1-\varphi_i) +$$
$$(1-\varphi_1)\dots(1-\varphi_i)\varphi_{i+1} =$$

$$= 1 - (1-\varphi_1)\dots(1-\varphi_i)(1-\varphi_{i+1}).$$

This proves our claim.

In particular,

$$\boxed{\begin{aligned}&\psi_1 + \psi_2 + \dots + \psi_m = \\ &= 1 - (1-\varphi_1)(1-\varphi_2)\dots(1-\varphi_m).\end{aligned}}$$

Let $x \in K$. Then, $x \in B_1(x_i)$ for some $1 \le i \le m$. Therefore, we have $\varphi_i(x) = 1$. It follows that

$$\psi_1(x) + \psi_2(x) + \dots + \psi_m(x) = 1$$

from the above formula.

This proves the property (c).