Let $f(x_1, \dots, x_n) =$

$$f_1(x_1) f_2(x_2) \dots f_n(x_n)$$

where $f_i : [a_i, b_i] \to \mathbb{R}$

are continuous functions.

Then f is continuous on I^n.

Moreover, we have

$$\int_{a_i}^{b_i} f(x_1, \dots, x_n)\, dx_i =$$

$$= \int_{a_i}^{b_i} f_1(x_1) \dots f_i(x_i) \dots f_n(x_n)\, dx_i =$$

$$= f_1(x_1) \dots f_{i-1}(x_{i-1}) f_{i+1}(x_i) \dots f_n(x_n) \cdot$$

$$\cdot \int_{a_i}^{b_i} f_i(x_i)\, dx_i$$

This implies that

$$I_\pi(f) = \int_{a_1}^{b_1} f(x_1)\, dx_1 \dots \int_{a_n}^{b_n} f(x_n)\, dx_n$$

Hence, for any two permutations π, π' we have

$$I_\pi(f) = I_{\pi'}(f)$$

for all special functions f.

Let $\mathcal{A}$ be the subalgebra of $C(I^n)$ spanned by functions $f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$, $f_i \in C([a_i, b_i])$, $1 \le i \le n$.

Then f contains constants.

Moreover, if $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are two different points in I^n, there exists $1 \le i \le n$, such that $x_i \neq y_i$

Let $\varphi : [a_i, b_i] \longrightarrow R$

$\varphi(x_i) \neq \varphi(y_i)$.

Then $f(x_1, \dots, x_n) = \varphi(x_i)$ is in $\mathcal{A}$ and

$$f(x_1, \dots, x_n) \neq f(y_1, \dots, y_n).$$

Hence, $\mathcal{A}$ differs points in $\mathcal{A}$.

By Stone - Weierstrass theorem $\mathcal{A}$ is dense in $C(I^n)$.

Put

$$vol(I^n) = |b_1 - a_1| \dots |b_n - a_n|.$$

Then we have

$$|I_\pi(f)| \leq$$

$$\int_{a_{\pi(n)}}^{b_{\pi(n)}} \dots \left(\int_{a_{\pi(1)}}^{b_{\pi(1)}} |f(x_1,\dots,x_n)| \, dx_{\pi(1)} \right) \dots dx_{\pi(n)} \leq$$

$$\|f\| \cdot vol(I^n).$$

Let $f \in C(I^n)$, and $\varepsilon > 0$.
Then there exists $g \in \mathcal{A}$ such that $\|f - g\| < \varepsilon$.
This implies that

$$|I_\pi(f) - I_\pi(g)| = |I_\pi(f-g)|$$
$$\leq \|f-g\| \cdot vol(I^n) < \varepsilon \cdot vol(I^n)$$

for any permutation π.
Let π and π' be two permutations. Then

$$|I_\pi(f) - I_\pi(g)| \leq \varepsilon \cdot \mathrm{vol}(I^n)$$

and

$$|I_{\pi'}(f) - I_{\pi'}(g)| \leq \varepsilon \cdot \mathrm{vol}(I^n).$$

Since $g \in \mathcal{A}$

$$I_\pi(g) = I_{\pi'}(g).$$

Hence, we have

$$|I_\pi(f) - I_{\pi'}(f)| =$$

$$|I_\pi(f) - I_\pi(g) + I_{\pi'}(g) - I_{\pi'}(f)|$$

$$\leq |I_\pi(f) - I_\pi(g)| + |I_{\pi'}(f) - I_{\pi'}(g)|$$

$$\leq 2 \cdot \varepsilon \cdot \mathrm{vol}(I^n).$$

It follows that

$$I_\pi(f) = I_{\pi'}(f)$$

for any $f \in C(I^n)$.

Therefore, linear forms I_π are all equal on $C(I^n)$.
We define
$$I(f) = I_\pi(f)$$
for any permutation π.
Hence I is an iterated integral and it doesn't depend on order of integration!