We want to define a linear form on $C_0(\mathbb{R}^n)$

$$f \longmapsto \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dx_1 \dots dx_n.$$

– the <u>integral</u> of f.

An n-<u>cell</u> I^n is a product

$$[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

of closed intervals $[a_i, b_i]$, $1 \le i \le n$, i.e.

$$I^n = \{(x_1, x_2, \dots, x_n) \mid a_i \le x_i \le b_i;\ 1 \le i \le n\}.$$

Since support of f is compact, it is a bounded set in $\mathbb{R}^n$ and contained in some n-cell I^n.

So we shall define

$$\int_{\mathbb{R}^n} f = \int_{I^n} f \, ,$$

Therefore, we have to define $\int_{I^n} f$ for all continuous functions on I^n.

Let $1 \le i \le n$. Then

$$x_i \longmapsto f(x_1, \dots, x_i, \dots, x_n)$$

is a continuous function on $[a_i, b_i]$, so we can define Riemann integral

$$\int_{a_i}^{b_i} f(x_1, \dots, x_i, \dots, x_n) \, dx_i$$

which is a function of $x_1, \dots, x_{i-1}, x_{i+1}, \dots x_n$.

We define

$$f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) = \int_{a_i}^{b_i} f(x_1,\dots,x_i,\dots,x_n)\,dx_i$$

<u>Claim</u>. f_i, $1\le i\le n$, are continuous functions on $[a_1,b_1]\times\dots\times[a_{i-1},b_{i-1}]\times[a_{i+1},b_{i+1}]\times\dots\times[a_n,b_n]$.

<u>Proof</u>. We have

$$f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - f_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n) =$$

$$= \int_{a_i}^{b_i} \big(f(x_1,\dots,x_i,\dots x_n) - f(y_1,\dots,x_i,\dots y_n)\big)\,dx_i$$

Since I^n is compact, f is uniformly continuous on I^n. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) < \delta$$

$$\Rightarrow |f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)| < \varepsilon.$$

This implies that

$$d((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots y_n))$$

$$< \delta$$

$$\Rightarrow d((x_1, \ldots, x_n), (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)) < \delta$$

$$\Rightarrow |f(x_1, \ldots, x_i, \ldots, x_n) - f(y_1, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots x_n)|$$

$$< \varepsilon$$

$$\Rightarrow |f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) -$$

$$- f_i(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)| \leq$$

$$\leq \int_{a_i}^{b_i} |f(x_1,\dots,x_i,\dots x_n) - f(y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_n)|\, dx_i$$

$$\leq \varepsilon\, |b_i - a_i|.$$

This proves the claim.

Let π be a permutation of $(1,2,\dots,n)$. Then, by induction we can evaluate

$$I_\pi(f) = \int_{a_{\pi(n)}}^{b_{\pi(n)}} \left(\int_{a_{\pi(n-1)}}^{b_{\pi(n-1)}} \left(\dots \int_{a_{\pi(1)}}^{\pi(n)} f(x_1,\dots,x_n)\, dx_{\pi(1)} \dots \right) dx_{\pi(n-1)} \right) dx_{\pi(n)}.$$

Clearly, $I_\pi : f \longmapsto I_\pi(f)$ is a linear form on $C(I)$.