X, Y metric spaces

d_X, d_Y metric

$f : X \longrightarrow Y$ is continuous if for any $x \in X$ and $\varepsilon > 0$ there exists $\delta_x > 0$ such that

$$d_X(x,y) < \delta_x \Longrightarrow d_Y(f(x), f(y)) < \varepsilon$$

$f : X \to Y$ is <u>uniformly continuous</u> if for any $\varepsilon > 0$ there exist $\delta > 0$ such that

$$d_X(x,y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \varepsilon.$$

f uniformly continuous $\Longrightarrow$ f is continuous

Converse is false ($e^x : \mathbb{R} \to \mathbb{R}$).

Theorem:
X compact metric space $f : X \to Y$ is a continuous function. Then f is uniformly continuous, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in X,\ d_X(x,y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Proof: Since $f : X \to Y$ is continuous, for any $\varepsilon > 0$ and $x \in X$ there exists $\delta_x > 0$ such that

$$d(x,y) < \delta_x \Longrightarrow d_Y(f(x), f(y)) < \varepsilon/2$$

Put $U_x = \{ y \mid d(x,y) < \frac{\delta_x}{2} \}$.
then $\{U_x ; x \in X\}$ is an open

cover of X. Let $x_1, \dots, x_n \in X$ be such that $\{U_{x_i} ; 1 \le i \le n\}$ is a finite subcover. Let $\delta_0 = \min \delta_{x_i}$. Take $\delta < \frac{\delta_0}{2}$.

Let $x \in X$. Then $x \in U_{x_i}$, hence

$$d(x, x_i) < \frac{\delta_{x_i}}{2},$$

Let $y \in K$ such that $d(x,y) < \delta$.

Then $d(y, x_i) \le d(y,x) + d(x, x_i) <$

$$< \delta + \frac{\delta_{x_i}}{2} < \frac{\delta_0}{2} + \frac{\delta_{x_i}}{2} \le \delta_{x_i}.$$

$$\Rightarrow \quad d_Y(f(x), f(x_i)) < \varepsilon/2$$

$$d_Y(f(y), f(x_i)) < \varepsilon/2$$

$$\Rightarrow \quad d_Y(f(x), f(y)) \le d_Y(f(x), f(x_i)) +$$

$$d_Y(f(x_i), f(y)) < \varepsilon. \qquad \blacksquare$$

Integration in $\mathbb{R}^n$

Let f be a function on a topological space X. Let

$\mathcal{U} = \{ U \text{ open } X,\ f|_U = 0 \}$

be the family of all open sets on which f restricts to 0.

Then the union of all elements in $\mathcal{U}$ is in $\mathcal{U}$, i.e., $\mathcal{U}$ contains a largest element with respect to the partial ordering given by inclusion.

Therefore, there exists the largest open set V

in X such that $f|_V = 0$.
The complement $X \setminus V$ is a closed set, which we call the <u>support</u> of f and denote by $\operatorname{supp} f$.

Let X be a Hausdorff space. Denote by $C_0(X)$ the set of all continuous functions $f : X \to \mathbb{R}$ such that $\operatorname{supp} f$ is compact.

<u>Claim</u>: $C_0(X)$ <u>is a vector space</u>.

$f \in C_0(X)$, $\alpha \in \mathbb{R}$, $\alpha f \in C_0(X)$.

$f, g \in C_0(X)$

$U = X \setminus (\operatorname{supp}(f) \cup \operatorname{supp}(g))$ $\quad x \in U$, $x \notin \operatorname{supp}(f)$

$x \notin \operatorname{supp}(g)$, $f(x) = 0$, $g(x) = 0 \Rightarrow$

$(f+g)(x) = f(x)+g(x) = 0$

$\Longrightarrow (f+g)|_U = 0$

$\mathrm{supp}(f+g) \subseteq X \setminus U = \mathrm{supp}(f) \cup \mathrm{supp}(g)$

compact

$\mathrm{supp}(f) \cup \mathrm{supp}(g)$ is compact

$\mathrm{supp}(f+g) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g)$

closed compact

$\Rightarrow \mathrm{supp}(f+g)$ is compact.