We proved last time that for a $n \times n$ matrix $T$

$$\|T - I\| < 1$$

implies that $T$ is invertible.

$GL(n, \mathbb{R}) = \{$all invertible matrices in $M_n(\mathbb{R})\}$

$M_n(\mathbb{R})$: $n \times n$ real matrices

$= \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$

$\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous

$\det^{-1}(\{0\})$ is a closed set in $M_n(\mathbb{R})$. $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

The unit ball with respect to

$\|.\|$ is in $GL(n, \mathbb{R})$.

$GL(n, \mathbb{R})$ is a group with respect to matrix multiplication.

$GL(n, \mathbb{R})$ inherits topology from $M_n(\mathbb{R})$.

$GL(n, \mathbb{R})$ is a <u>topological group</u>.

This is also an example of a Lie group since the multiplication is a differentiable map.

<u>Claim</u>, For $x \in U$, $F'(x)$ is invertible.

<u>Proof</u> (of the claim): $U$ was a ball centered at $a$ such that

$$\|A - F'(x)\| \leq \frac{1}{2\|A^{-1}\|}.$$

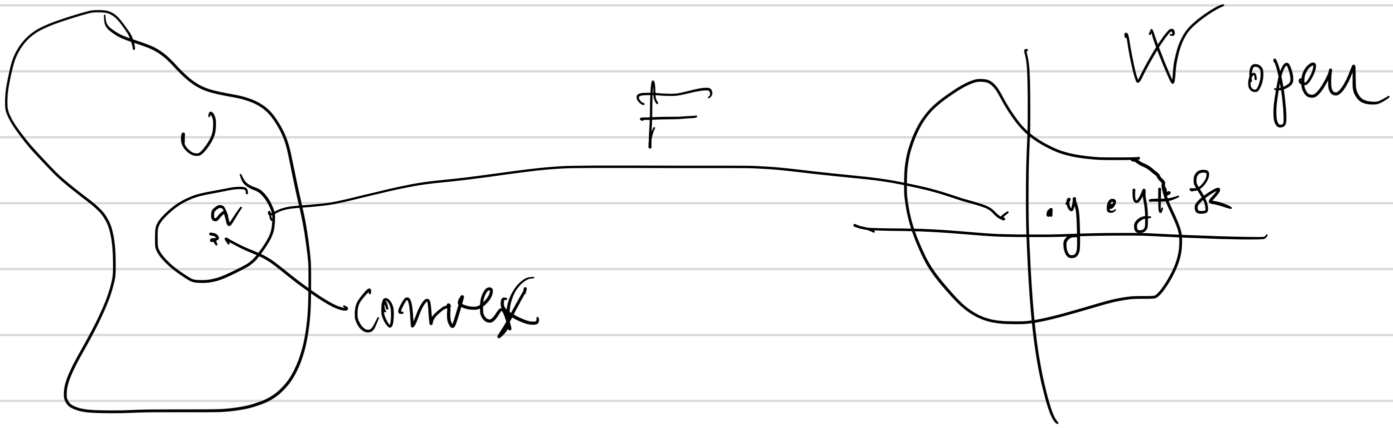
Therefore

$$\|I - A^{-1}F'(x)\| = \|A^{-1}(A - F'(x))\| \leq$$

$$\leq \|A^{-1}\| \cdot \|A - F'(x)\| \leq \frac{1}{2} < 1$$

Hence, $A^{-1}F'(x) \in GL(n, \mathbb{R})$.

Since $A$ is invertible, the product $A \cdot (A^{-1}F'(x)) = F'(x)$ is invertible. $\boxtimes$




$F: U \to W$ is a bijection of open sets

Pick $y, y+k \in W$. $\exists\, x, x+h \in U$ such that $F(x) = y$, $F(x+h) = y+k$.

$$\varphi(x+h) - \varphi(x) = (x+h) + A^{-1}\big(y - \underbrace{F(x+h)}_{\overset{\shortparallel}{y+k}}\big) - x - A^{-1}\big(\underbrace{y - F(x)}_{=0}\big) =$$

$$= h + A^{-1}(\cancel{y} - \cancel{y} - k) = h - A^{-1}k.$$

$$\| h - A^{-1}k \| = \| \varphi(x+h) - \varphi(x) \| \le \tfrac{1}{2} \| h \|$$

$$\left\| h - A^{-1}k \right\| \le \frac{1}{2}\|h\|$$

$$\Big|\, \|h\| - \|A^{-1}k\| \,\Big| \le \|h - A^{-1}k\| \le \frac{1}{2}\|h\|$$

$$\|h\| - \|A^{-1}k\| \le \frac{1}{2}\|h\|$$

$$\Rightarrow \quad -\|A^{-1}k\| \le -\frac{1}{2}\|h\|$$

$$\Rightarrow \quad \boxed{\|A^{-1}k\| \ge \frac{1}{2}\|h\|}$$

$$\Rightarrow \quad \frac{1}{2}\|h\| \le \|A^{-1}k\| \le \|A^{-1}\| \cdot \|k\|$$

if $k \neq 0$

$$\boxed{\frac{\|h\|}{\|k\|} \le 2\|A^{-1}\|}$$

Let $T = F'(x)^{-1}$.

$$G(y+k) - G(y) - Tk =$$

$$\cancel{x} + h - \cancel{x} - Tk = h - Tk =$$

$$= T(T^{-1}h - k) = T(F'(x)h - k)$$

$$= T\big(F'(x)h - (F(x+h) - F(x))\big) =$$

$$= -T\big(F(x+h) - F(x) - F'(x)h\big).$$

$$\frac{\|G(y+k) - G(y) - Tk\|}{\|k\|} \leq$$

$$\leq \|T\| \cdot \frac{\|F(x+h) - F(x) - F'(x)h\|}{\|h\|} \cdot \frac{\|h\|}{\|k\|} \leq$$

$$\leq 2\|T\| \cdot \|A^{-1}\| \underbrace{\frac{\|F(x+h) - F(x) - F'(x)h\|}{\|h\|}}_{\to 0}$$

as $k \to 0$.

$G$ is differentiable, $G'(y) = T$. ∎