We proved that $F|_U$ is 1-to-1.

Let $W = f(U)$. Take y_0 in W and $x_0 \in U$ such that $F(x_0) = y_0$.

Let B be a ball around x_0 such that $\overline{B} \subset U$ of radius ε.

$\overline{B}$ is a closed subset of $\mathbb{R}^n$.

We proved that $\mathbb{R}^n$ is complete.

Let $(x_n, n \in \mathbb{N})$ be a Cauchy sequence in $\overline{B}$. It is a Cauchy sequence in $\mathbb{R}^n$.

It converges in $\mathbb{R}^n$. Since $\overline{B}$ is closed, the limit is in $\overline{B}$ (by homework).

$\overline{B}$ is complete!

Fix $\delta > 0$ and consider y, $\|y - y_0\| < \delta$.

$$\varphi(x_0) = x_0 + A^{-1}(y - \underbrace{y_0}_{= F(x_0)})$$

$$\|\varphi(x_0) - x_0\| = \|A^{-1}(y - y_0)\| \leq \|A^{-1}\| \|y - y_0\|$$

$$\Rightarrow \|\varphi(x_0) - x_0\| \leq \|A^{-1}\| \cdot \|y - y_0\| < \|A^{-1}\| \cdot \delta$$

$$\Rightarrow \|\varphi(x) - x_0\| \leq \|\varphi(x) - \varphi(x_0)\| + \|\varphi(x_0) - x_0\|$$

$$\leq \frac{1}{2}\|x - x_0\| + \|A^{-1}\| \delta = \underline{\frac{1}{2}\varepsilon + \|A^{-1}\| \cdot \delta}$$

If δ is small $\Rightarrow$ $r \leq \varepsilon$

$\varphi(x) \in \overline{B}$. $\varphi : \overline{B} \to \overline{B}$ contraction

$\Rightarrow$ has a fixed point $x \in \overline{B}$.

$\Rightarrow$ for this x, $F(x) = y$. Hence y is in the image of F.

It follows that the ball $\{y \mid \|y - y_0\| < \delta\}$ of radius δ centered at y_0 is in W.

Since y_0 was arbitrary, W is an open set.

It follows that $F(U) = W$ is an open set in $\mathbb{R}^n$. $F : U \longrightarrow W$ is a bijection. Has an inverse $G : W \longrightarrow U$.

It remains to show that G is continuously differentiable.

<u>We return to Linear Algebra</u>:

$M_n(\mathbb{R})$ $n \times n$ matrices

$\|.\|$ operator norm on $M_n(\mathbb{R})$

$\|A \cdot B\| \leq \|A\| \cdot \|B\|$

By induction
$$\|A^n\| \leq \|A\|^n, \quad n \in \mathbb{N}.$$

<u>Claim</u>: $T \in M_n(\mathbb{R})$, $\|T-I\| < 1$.
Then T is invertible.

<u>Proof</u>. We claim that
$$T^{-1} = \underbrace{\sum_{n=0}^{\infty} (I-T)^n}_{\text{convergent}}$$

$$S_N = \sum_{n=0}^{N} (I-T)^n$$

$$|(S_N)_{ij}| = \left|\sum_{n=0}^{N} ((I-T)^n)_{ij}\right| \leq$$
$$\leq \sum_{n=0}^{N} |((I-T)^n)_{ij}|$$

Since
$$\max_{i,j} |A_{ij}| \leq \|A\|$$

we see that

$$\sum_{n=0}^{N} \left|\left((I-T)^n\right)_{ij}\right| \le \sum_{n=0}^{N} \|I-T\|^n \le \sum_{n=0}^{N} g^n$$

if $\|I-T\| = g$. Since $g<1$

$$\sum_{n=0}^{N} g^n \longrightarrow \frac{1}{1-g}$$

Hence $\displaystyle\sum_{n=0}^{\infty} \left|\left((I-T)^n\right)_{ij}\right|$ converges.

$$\Longrightarrow \quad \sum_{n=0}^{\infty} \left((I-T)^n\right)_{ij} \text{ converges}$$

$$S_N \longrightarrow S = \sum_{n=0}^{\infty} (I-T)^n$$

$$(I-T)S = \sum_{n=1}^{\infty} (I-T)^n = S - I$$

$$\cancel{S} - TS = \cancel{S} - I \qquad TS = I$$

analogously, $ST = I$.

T is invertible and S is the inverse of T. This proves the claim.