Theorem. Let O be an open set in $\mathbb{R}^n$ and $F: O \to \mathbb{R}^n$ a continuously differentiable function. Assume that $F'(a)$, $a \in O$, is invertible. Then there exist open neighborhoods U of a and V of $b = F(a)$ such that

(i) $F: U \to V$ is a bijection;

(ii) the inverse function $G: V \to U$ is continuously differentiable at

$$\boxed{G'(b) = F'(a)^{-1}.}$$

First we need some properties of operator norm.

$$Ae_j = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_j} = \begin{bmatrix} A_{ij} \\ \vdots \\ \vdots \\ A_{mj} \end{bmatrix}$$

$$\|Ae_j\|^2 \le \|A\|^2 \|e_j\|^2 = \|A\|^2$$

$$\|Ae_j\|^2 = \sum_{i=1}^{m} |A_{ij}|^2 \le \|A\|^2$$

$$\Rightarrow |A_{ij}|^2 \le \|A\|^2 \Rightarrow |A_{ij}| \le \|A\|$$

$$\boxed{M = \max_{i,j} |A_{ij}| \le \|A\|}$$

$$\|Ax\|^2 = \sum_{i=1}^{m} (Ax)_i^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ji} x_j \right)^2$$

$$\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |A_{ji}| \cdot |x_j| \right)^2 \leq M^2 \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |x_j| \right)^2$$

$$\leq M^2 \sum_{i=1}^{m} n \cdot \sum_{j=1}^{n} |x_j|^2 = n^2 M^2 \|x\|^2$$

by Cauchy-Schwarz

$$\Rightarrow \quad \|Ax\|^2 \leq n^2 M^2 \|x\|^2$$

$$\Rightarrow \quad \|Ax\| \leq n \cdot M \cdot \|x\|$$

$$\Rightarrow \quad \|A\| \leq n \cdot M$$

$$\boxed{M \leq \|A\| \leq n \cdot M}$$

If $x \longmapsto A(x)$ is continuous, for $\varepsilon > 0$, $\exists \delta > 0$, $|A(x)_{ij} - A(x_0)_{ij}| < \varepsilon$ as $\|x - x_0\| < \delta \Rightarrow \|A(x) - A(x_0)\| < n \cdot \varepsilon$

$x \longmapsto \|B(x)\|$ is continuous.

<u>Proof.</u> $A = F'(a)$.

Consider, for $y \in \mathbb{R}^n$, the function

$$\varphi(x) = x + A^{-1}(y - F(x))$$

$$\varphi(x) = x \Leftrightarrow A^{-1}(y - F(x)) = 0 \Leftrightarrow$$

$$y - F(x) = 0 \Leftrightarrow y = F(x).$$

$$\varphi'(x) = I - A^{-1}F'(x) =$$

$$= A^{-1}(A - F'(x)).$$

$$\|\varphi'(x)\| \leq \|A^{-1}\| \cdot \|A - F'(x)\|$$

Since $F'(x)$ is continuous on O,

there exists an open ball U centered at a

such that $\|\underset{\substack{\| \\ F'(a)}}{A} - F'(x)\| \leq \frac{1}{2\|A^{-1}\|}$,

$$\Rightarrow \|\varphi'(x)\| \leq \|A^{-1}\| \cdot \frac{1}{2\|A^{-1}\|}.$$

Since U is convex, we have
$$\|\varphi(c)-\varphi(d)\| \le \frac{1}{2}\|c-d\|.$$
for $c, d \in U$. Hence, φ is a contraction. Hence, it can have at most one fixed point in U.

So, for $y \in \mathbb{R}^n$, the equation $y=F(x)$ has at most one solution in U.

$\Rightarrow F: U \longrightarrow \mathbb{R}$ is 1-to-1 (i.e. an injection).