<u>Theorem</u>. Equivalent

(i) $F$ is continuously differentiable on $U$

(ii) $\partial_i F_j$, $1 \le i \le n$, $1 \le j \le m$, exist and are continuous on $U$.

<u>Proof</u>. We proved (i) $\Longrightarrow$ (ii).

Converse (ii) $\Longrightarrow$ (i):

Assume that $\partial_i F_j$ exist and are continuous on $U$.

① $F = (F_1, \dots, F_m)$. It is enough to show that $F_j$ are continuously differentiable

$$\|F(x_0+h) - F(x_0) - Ah\|^2 \le$$

$$\sum_{i=1}^{m} \left| F_i(x_0+h) - F_i(x_0) - \sum_{j=1}^{n} A_{ji} h_j \right|^2$$

$$\frac{\|F(x_0+h)-F(x_0)-Ah\|^2}{\|h\|^2} =$$

$$= \sum_{i=1}^{n} \frac{|F_i(x_0+h)-F_i(x_0)-\sum A_{ji}h_j|^2}{\|h\|^2} \to 0$$

Since $F_j$ are differentiable at $x_0$.

$\Rightarrow$ $F$ is differentiable at $x_0$

$$F'(x_0) = \left(\partial_i F_j(x_0)\right) \Longrightarrow F' \text{ is continuous}$$

continuous

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Can assume that $m=1$.

$$F(x_0+h)-F(x_0) = \sum_{j=1}^{n} F(x_0+v_j) - F(x_0+v_{j-1})$$

$$h=(h_1,\dots,h_m)$$

$$v_k = \sum_{i=1}^{k} h_i e_i \quad v_n = h, \quad v_1 = (h_1, 0, \dots, 0)$$

$$F(x_0+h)-F(x_0)=\sum_{k=1}^{n}\left(F(x_0+v_k)-F(x_0+v_{k-1})\right)=\sum_{k=1}^{n} h_k\,\partial_k F(x_0+v_{k-1}+\vartheta_k h_k e_k)$$ for some $0\le\vartheta_k\le 1$

– by mean value theorem.

Can assume that

$$\left|\partial_j F(x_0+h)-\partial_j F(x_0)\right|<\frac{\varepsilon}{n}$$

ball $\|x_0\|$

$$\left|F(x_0+h)-F(x_0)-\sum_{k=1}^{n}\partial_k F(x_0)h_k\right|\le$$

$$\le\sum_{k=1}^{n}\left|\left(\partial_k F(x_0+v_{k-1}+\vartheta_k h_k e_k)-\partial_k F(x_0)\right)h_k\right|$$

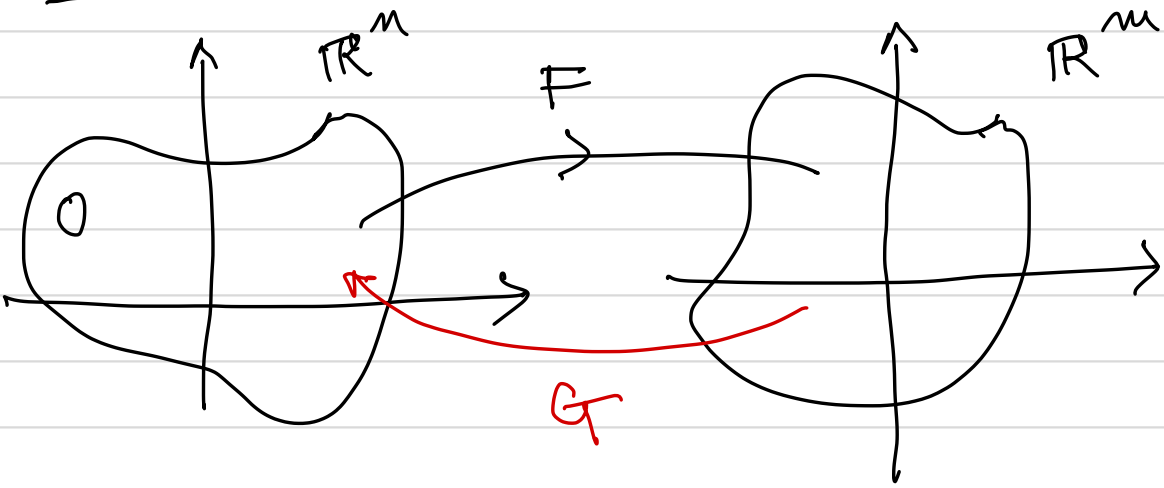
$$\le\sum_{k=1}^{n}\frac{\varepsilon}{n}\cdot|h_k|=\frac{\varepsilon}{n}\cdot\sum_{k=1}^{n}|h_k|\le$$

$$\frac{\varepsilon}{n}\cdot\sqrt{n}\cdot\|h\|=\frac{\varepsilon}{\sqrt{n}}\cdot\|h\|$$

by Cauchy-Schwarz inequality

F is differentiable! ∎

# Inverse function theorem




$F$ and $G$ are differentiable

and $G = F^{-1}$, i.e.

$$(G \circ F)(x) = x$$

$\Rightarrow$ By the chain rule

$$G'(F(x_0)) \circ F'(x_0) = I.$$

If $y = F(x)$, $(F \circ G)(y) = y$

$$\Rightarrow F'(G(y_0)) \circ G'(y_0) = I$$

$F'(x_0)$ is an invertible linear map

and $\boxed{n = m.}$ $\quad F'(x_0)^{-1} = G'(F(x_0))$.

<u>Theorem</u>. Let $O$ be an open set in $\mathbb{R}^n$ and $F: O \longrightarrow \mathbb{R}^n$ a continuously differentiable function. Assume that $F'(a)$, $a \in O$, is invertible. Then there exist open neighborhoods $U$ of $a$ and $V$ of $b = F(a)$ such that

(i) $F: U \longrightarrow V$ is a bijection;

(ii) the inverse function $G: V \longrightarrow U$ is continuously differentiable at

$$\boxed{G(b) = F(a)^{-1}.}$$