$$F(x) = (F_1(x), F_2(x), \dots, F_m(x))$$

Let $h_j = te_j$. Then

$$\lim_{t\to 0} \frac{\|F(x+h_j) - F(x) - F'(x)h_j\|}{|t|} = 0$$

$F(x+te_j) - F(x) - tF'(x)e_j$ is

a vector with coordinates

$$F_i(x+te_j) - F_i(x) - F'(x)_{ij}t$$

since

$$F'(x)e_j = \begin{bmatrix} F'(x)_{11} & \cdots & F'_{1n} \\ \vdots & & \vdots \\ F'(x)_{m1} & \cdots & F'(x)_{mn} \end{bmatrix} \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \text{-}j \text{ row} \\ \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} F'_{1j}(x) \\ \vdots \\ \\ F'_{mj}(x) \end{bmatrix}$$

Hence,

$$\lim_{t\to 0} \frac{\left|F_i(\overbrace{x+te_j}^{(x_1,\ldots,x_j+t,\ldots x_n)}) - F_i(x) - F'(x)_{ij}\, t\right|}{|t|} = 0$$

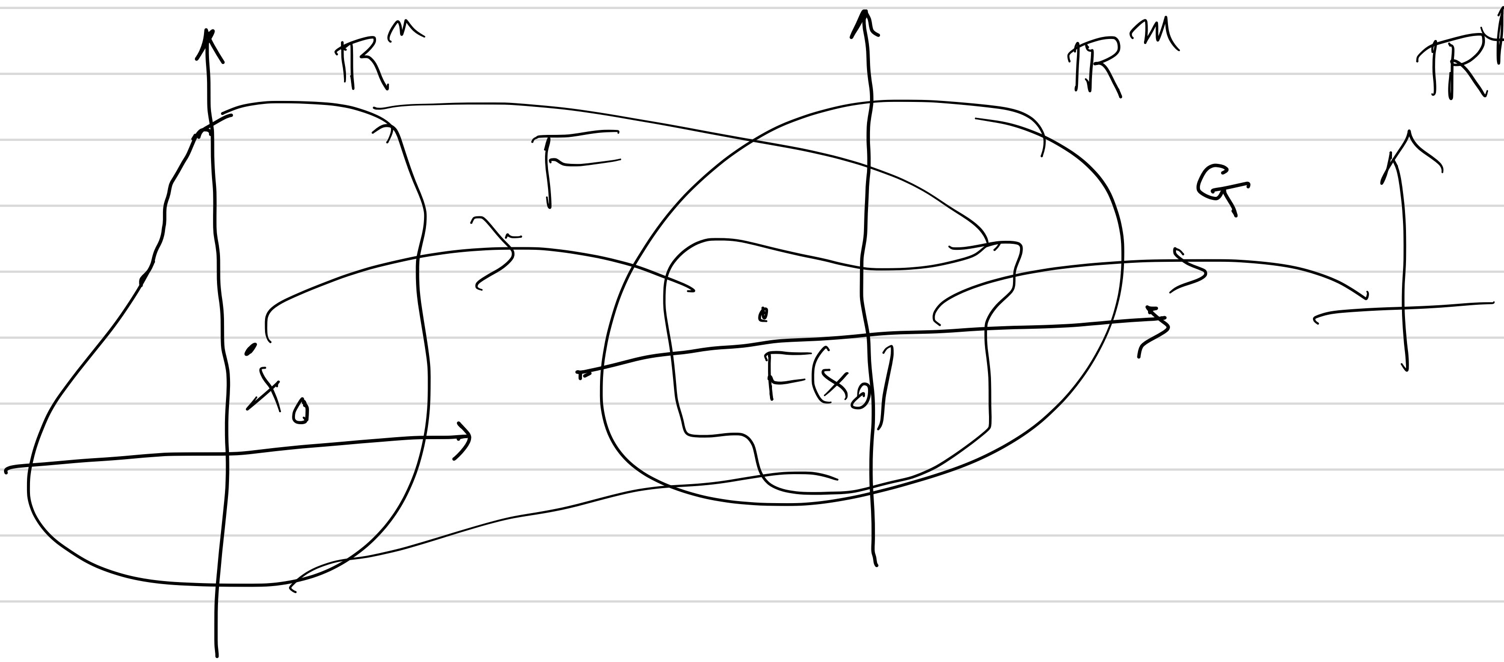
$$\Rightarrow \quad \boxed{F'(x)_{ij} = \frac{\partial F_i(x)}{\partial x_j}}$$

partial derivatives

$$F'(x) = \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m(x)}{\partial x_1} & \cdots & \frac{\partial F_m(x)}{\partial x_n} \end{bmatrix}$$

matrix of the derivative of F at x.

## Chain rule




$$H = G \circ F$$

Assume that $F$ is differentiable at $x_0$, $G$ is differentiable at $F(x_0)$. Then $H$ is differentiable at $x_0$ and

$$H'(x_0) = G'\left[F(x_0)\right) \cdot F'(x_0),$$

Proof:

$$\frac{\|F(x_0+h) - F(x_0) - Ah\|}{\|h\|} \to 0$$

$$\frac{\| G(y_0+k) - G(y_0) - Bk \|}{\|k\|} \to 0$$

$$\| G(F(x_0+h)) - G(\underset{\overset{\shortparallel}{y_0}}{F(x_0)}) -$$

$$B(F(x_0+h) - F(x_0)) \| \le \underset{\overset{\downarrow}{0}}{\omega(h)} \| F(x_0+h) -$$

$$F(x_0) \|$$

$$\| F(x_0+h) - F(x_0) \| \le (\varepsilon + \|A\|) \|h\|$$

as $\|h\| < \delta$

$$\| G(F(x_0+h)) - G(F(x_0)) -$$

$$B(F(x_0+h) - F(x_0)) \| \le \omega(h) (\varepsilon + \|A\|) \cdot \|h\|$$

On the other hand, we have

$$\|B(F(x_0+h)-F(x_0))-BAh\| \leq$$

$$\leq \|B\| \cdot \|F(x_0+h)-F(x_0)-Ah\| \leq$$

$$\|B\| \cdot \underset{\searrow 0}{\omega_1(h)} \|h\|$$

$$\|G(F(x_0+h))-G(F(x_0))-BAh\| =$$

$$= \|\underline{G(F(x_0+h))-G(F(x_0))-B(F(x_0+h)-F(x_0))} +$$

$$+ \underline{B(F(x_0+h)-F(x_0))-BAh}\| \leq$$

$$\underset{\searrow 0}{\omega(h)} (\varepsilon+\|A\|)\|h\| + \|B\| \cdot \underset{\searrow 0}{\omega_1(h)} \|h\|$$

By division with $\|h\|$ we get

$$\frac{\|G(F(x_0+h))-G(F(x_0))-BAh\|}{\|h\|} \to 0$$

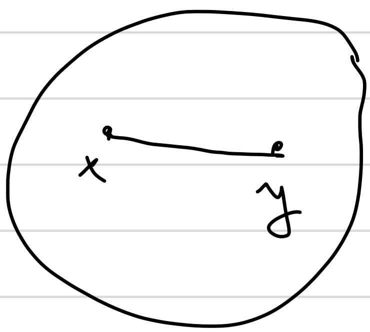
as $\|h\| \to 0$.

$\blacksquare$

Let $U \subset \mathbb{R}$ be an open convex set

(i.e. $x, y \in U \Rightarrow tx + (1-t)y \in U$

for $t \in [0,1]$),




Assume that

$f : U \to \mathbb{R}^m$ is

differentiable and

$\|f'(x)\| \leq M$ for all $x \in U$,

<u>Theorem</u>

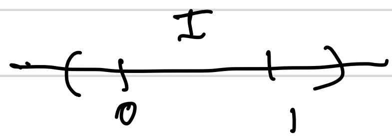
$$\|f(a) - f(b)\| \leq M \|a - b\|$$

for all $a, b \in U$.

<u>Proof</u>: Let $\gamma(t) = (1-t)a + tb$

Then $\gamma(0) = a$, $\gamma(1) = b$.

$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ is differentiable.




$\gamma(I) \subset U$ since $U$ is convex

$I$ is open interval

$$g = f \circ \gamma : I \longrightarrow \mathbb{R}^m$$

is differentiable

$$g'(t) = f'(\gamma(t)) \circ \gamma'(t) =$$

$$= f'(\gamma(t)) \cdot (b-a)$$

$$\|g'(t)\| = \|f'(\gamma(t))\| \cdot \|b-a\| \leq M \cdot \|b-a\|$$

for $t \in I$.

Let $y \in \mathbb{R}^m$. Then

$$\varphi(t) = (g(t)|y) \quad \text{is a}$$

differentiable real function on $I$.

We have

$$\varphi'(t) = (g'(t)\,|\,y)$$

for $t \in I$. By mean value theorem

$$\varphi(1) - \varphi(0) = \varphi'(c)$$

for some $c \in [0,1]$.

$$|\varphi(1) - \varphi(0)| \leq |(y\,|\,g'(c))| \leq$$
$$\|y\| \cdot \|g'(c)\| \leq M\,\|y\| \cdot \|b-a\|$$

Put $y = g(1) - g(0)$. Then

$$\varphi(1) - \varphi(0) = (g(1) - g(0)\,|\,g(1) - g(0))$$

$$|\varphi(1) - \varphi(0)| = \|g(1) - g(0)\|^2 \leq M \cdot \|g(1) - g(0)\| \cdot$$
$$\|b-a\|$$

$$\Rightarrow \underbrace{\|g(1) - g(0)\|}_{=\,\|f(b) - f(a)\|} \leq M \cdot \|b-a\|$$

$\|f(b) - f(a)\|$. ▨

Cor. Let $U$ be convex & open $f : U \longrightarrow \mathbb{R}^m$. If $f'(x) = 0$ for all $x \in U \implies f$ is constant.

$\boxtimes$

Let $U$ be an open set in $\mathbb{R}^n$, $F : U \longrightarrow \mathbb{R}^m$. $F$ is continuously differentiable on $U$ if it is differentiable at any point of $U$ and $x \longmapsto F'(x)$ is continuous on $U$.

At $x_0 \in U$, $(F'(x_0 + te_i) \mid f_j) = \partial_i F_j(x_0)$

diff. and continuous.

$\implies F$ cont. differentiable

$\implies \partial_i F_j$ are cont.

**Theorem.** Equivalent

(i) $F$ is cont. diff. on $U$

(ii) $\partial_i F_j$, $1 \le i \le n$, $1 \le j \le m$, exist and are continuous on $U$.

**Proof.** We proved (i) $\Longrightarrow$ (ii).

Converse (ii) $\Longrightarrow$ (i)

Assume that $\partial_i F_j$ exist and are continuous on $U$.

(1) $F = (F_1, \dots, F_m)$. It is enough to show that $F_j$ are cont. diff.

$$\|F(x_0+h) - F(x_0) - Ah\|^2 \le$$
$$\sum_{i=1}^{m} \Big| F_i(x_0+h) - F_i(x_0) - \sum_{j=1}^{n} A_{ji} h_j \Big|^2$$

$$\frac{\|F(x_0+h)-F(x_0)-Ah\|^2}{\|h\|^2} =$$

$$= \sum_{i=1}^{n} \frac{|F_i(x_0+h)-F_i(x_0)-\sum A_{ji}h_j|^2}{\|h\|^2} \to 0$$

Since $F_j'$ are diff. at $x_0 \Longrightarrow F$ is diff. at $x_0$

$$F'(x_0) = \left(\underbrace{\partial_i F_j(x_0)}_{\text{cont.}}\right) \Longrightarrow F' \text{ is cont.}$$

---

Can assume that $m=1$.

$$F(x_0+h)-F(x_0) = \sum_{j=1}^{n} F(x_0+v_j)-F(x_0+v_{j-1})$$

$$h=(h_1,\ldots,h_n)$$

$$v_k = \sum_{i=1}^{k} h_i e_i$$

$$F(x_0+h) - F(x_0) = \sum_{j=1}^{n} h_j \partial_j F(x_0 + v_{k-1} + \vartheta_k h_k e_k)$$

$0 \le \vartheta_k \le 1$.

by mean value theorem

Can assume that

$$\forall (x_0) \quad |\partial_j F(x_0+h) - \partial_j F(x_0)| < \frac{\varepsilon}{n}$$

$$|F(x_0+h) - F(x_0) - \sum \partial_j F(x_0) h_j| \le$$

$$\le \sum_{j=1}^{n} \underbrace{|\partial_j F(x_0 + v_{k-1} + \vartheta_k h_k e_k) - \partial_j F(x_0))}_{\le \varepsilon/n} h_j|$$

$$\le \frac{\varepsilon}{n} \cdot \sum_{j=1}^{n} |h_j| \le \frac{\varepsilon}{\sqrt{n}} \cdot \|h\|$$

Cauchy Schwartz

$\Rightarrow F$ is differentiable.

$\blacksquare$