Let $h_j = te_j$. Then
\[
\lim_{t \to 0} \frac{\|F(x+h_j) - F(x) - F'(x)h_j\|}{|t|} = 0
\]

$F(x+te_j) - F(x) = tF'(x)e_j$ is a vector with coordinates $F_i(x+te_j) - F_i(x) - F'(x)_i t$

Since
\[
F'(x)e_j = \begin{bmatrix} F'(x)_1 & \cdots & F'(x)_m \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{-j \text{ row}}
\]

$n \begin{bmatrix} F'_{ij}(x) \\ \vdots \\ F'_{mj}(x) \end{bmatrix}$
Hence, \[
\lim_{t \to 0} \frac{F_i(x + te_j) - F_i(x) - F'(x)_i \cdot t}{|t|} = 0
\]

\[
\Rightarrow F'(x)_{ij} = \frac{\partial F_i(x)}{\partial x_j}
\]

partial derivatives

\[
F'(x) = \begin{bmatrix}
\frac{\partial F_1(x)}{\partial x_1} & \cdots & \frac{\partial F_1(x)}{\partial x_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_m(x)}{\partial x_1} & \cdots & \frac{\partial F_m(x)}{\partial x_m}
\end{bmatrix}
\]

matrix of the derivative of \( F \) at \( x \).
**Chain rule**

Assume that $F$ is differentiable at $x_0$, $G$ is differentiable at $F(x_0)$. Then $H$ is differentiable at $x_0$ and

$$H'(x_0) = G' (F(x_0)) \cdot F'(x_0),$$

**Proof:**

$$\lim_{\|h\| \to 0} \frac{\| F(x_0 + h) - F(x_0) - A h \|}{\| h \|} = 0$$
\[
\frac{\| G(y_0 + h) - G(y_0) - B \|}{\| h \|} \rightarrow 0
\]

\[
\| G(F(x_0 + h)) - G(F(x_0)) - \| \leq \omega(h) \| F(x_0 + h) - F(x_0) \|
\]

\[
\| F(x_0 + h) - F(x_0) \| \leq (\varepsilon + \| A \|) \| h \|
\]

As \( \| h \| < \delta \)

\[
\| G(F(x_0 + h)) - G(F(x_0)) - B(F(x_0 + h) - F(x_0)) \| \leq \omega(h) (\varepsilon + \| A \|) \| h \|
\]

On the other hand, we have
\begin{align*}
&\|B(F(x_0+h) - F(x_0)) - BAh\| \leq \\
&\leq \|B\| \cdot \|F(x_0+h) - F(x_0) - Ah\| \\
&\leq \|B\| \cdot \omega_1(h) \|h\| \\
&\to 0
\end{align*}

\begin{align*}
&\|G(F(x_0+h)) - G(F(x_0)) - BAh\| = \\
&= \|G(F(x_0+h)) - G(F(x_0)) - B(F(x_0+h) - F(x_0)) + \\
&B(F(x_0+h) - F(x_0)) - BAh\| \leq \\
&\leq \omega(h) \left( \varepsilon + \|Ah\| \right) \|h\| + \|B\| \cdot \omega_1(h) \|h\| \\
&\to 0
\end{align*}

By division with \(\|h\|\) we get
\[ \frac{\|G(F(x_0+h)) - G(F(x_0)) - BAh\|}{\|h\|} \to 0 \]

as \(\|h\| \to 0\).
Let $U \subset \mathbb{R}$ be an open convex set (i.e. $x, y \in U \Rightarrow tx + (1-t)y \in U$ for $t \in [0,1]$).

Assume that $f : U \rightarrow \mathbb{R}^n$ is differentiable and $\|f'(x)\| \leq M$ for all $x \in U$.

**Theorem**

$$\|f(a) - f(b)\| \leq M \|a - b\|$$

for all $a, b \in U$.

**Proof**: Let $\gamma(t) = (1-t)a + t b$

Then $\gamma(0) = a$, $\gamma(1) = b$. 
\( \gamma: \mathbb{R} \rightarrow \mathbb{R}^n \) is differentiable.

\[ \gamma'(t) \in \gamma((0,1)) \subset U \text{ since } U \text{ is convex} \]

\( I \) is open interval

\[ g = f \circ \gamma : I \rightarrow \mathbb{R}^n \]

is differentiable

\[ g'(t) = f'(\gamma(t)) \circ \gamma'(t) = f'(\gamma(t)) \cdot (b-a) \]

\[ \|g'(t)\| = \|f'(\gamma(t))\| \cdot \|b-a\| \leq M \cdot \|b-a\| \]

for \( t \in I \).

Let \( y \in \mathbb{R}^m \). Then

\[ \varphi(t) = (g(t) | y) \]

is a differentiable real function on \( I \).
We have

\[ \varphi'(t) = (g'(t) | y) \]

for \( t \in I \). By mean value theorem

\[ \varphi(1) - \varphi(0) = \varphi'(c) \]

for some \( c \in [0,1] \).

\[ |\varphi(1) - \varphi(0)| \leq |(g | g'(c))| \leq \|y\| \cdot \|g'(c)\| \leq M \|y\| \cdot \|b - a\| \]

Put \( y = g(1) - g(0) \). Then

\[ \varphi(1) - \varphi(0) = (g(1) - g(0) | g(1) - g(0)) \]

\[ |\varphi(1) - \varphi(0)| = \|g(1) - g(0)\| \leq M \|g(1) - g(0)\| \cdot \|b - a\| \]

\[ \Rightarrow \|g(1) - g(0)\| \leq M \|b - a\| \]

\[ \|f(b) - f(a)\| \]
Let $U$ be convex and open
$f : U \rightarrow \mathbb{R}^m$. If $f'(x) = 0$ for all $x \in U \Rightarrow f$ is constant.

Let $U$ be an open set in $\mathbb{R}^m$
$F : U \rightarrow \mathbb{R}^m$. $F$ is continuously differentiable on $U$ if it is differentiable at any point of $U$ and $x \mapsto F'(x)$ is continuous on $U$.

At $x_0 \in U$, $(F'(x_0 + te_i)|f_j) = \partial_i F_j(x_0)$
diff. and continuous.
$\Rightarrow F$ cont. differentiable
$\Rightarrow \partial_i F_j$ are cont.
**Theorem. Equivalent**

(i) $F$ is cont. diff. on $U$

(ii) $\partial_i F_j$, $1 \leq i \leq n$, $1 \leq j \leq m$, exist and are continuous on $U$.

**Proof.** We proved $(i) \Rightarrow (ii)$.

Converse $(ii) \Rightarrow (i)$

Assume that $\partial_i F_j$ exist and are continuous on $U$.

Let $F=(F_1, \ldots, F_m)$. It is enough to show that $F_j$ are cont. diff.

$$
\|F(x_0+h)-F(x_0)-Ah\| \leq \sum_{i=1}^{m} \left| F_i(x_0+h)-F_i(x_0)-\sum_{j=1}^{n} A_{ij} h_j \right|^2
$$
\[
\frac{\|F(x_0 + h) - F(x_0) - Ah\|^2}{\|h\|^2} = \\
= \sum_{i=1}^{m} \frac{|F_i(x_0 + h) - F_i(x_0) - \sum A_{ij}h_j|^2}{\|h\|^2} \\
\]

Since \( F_j \) are diff. at \( x_0 \), \( F \) is diff. at \( x_0 \)

\[
F'(x_0) = \left( \begin{array}{c}
\partial_i F_j(x_0) \\
\end{array} \right) \Rightarrow F' \text{ is cont. cont.}
\]

\[\text{Can assume that } m = 1, \]

\[
F(x_0 + h) - F(x_0) = \sum_{i=1}^{m} F(x_0 + \theta_j) - F(x_0 + \theta_{j-1})
\]

\[h = (h_1, \ldots, h_m)\]

\[\theta \preceq = \sum_{i=1}^{m} h_i e_i \]
\[ F(x_0 + h) - F(x_0) = \sum_{j=1}^{m} h_j \frac{\partial F}{\partial x_j} (x_0 + \eta_k h_k e_k) \]

\[ 0 \leq \eta_k \leq 1. \]

By the Mean Value Theorem, we can assume that

\[ |\frac{\partial F(x_0 + h) - \partial F(x_0)}{h_j}| < \frac{\varepsilon}{m} \]

\[ |F(x_0 + h) - F(x_0) - \sum_{j=1}^{m} \frac{\partial F}{\partial x_j} (x_0) h_j| \leq \sum_{j=1}^{m} \left| \frac{\partial F}{\partial x_j} (x_0 + \eta_k h_k e_k) - \frac{\partial F}{\partial x_j} (x_0) \right| h_j \]

\[ \leq \frac{\varepsilon}{m} \sum_{j=1}^{m} |h_j| \leq \frac{\varepsilon}{m} \|h\| \]

(Cauchy-Schwartz)

\[ \Rightarrow F \text{ is differentiable.} \]