Contraction principle

X complete metric space

$\varphi : X \longrightarrow X$ map.

φ is a contraction if there exists $0 < c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq c\, d(x, y)$$

for all $x, y \in X$.

Theorem. There exists a unique point $x_0 \in X$ such that $\varphi(x_0) = x_0$.

$\varphi(x_0) = x_0$ – x_0 is a fixed point for φ.

- there exists a fixed point

$\Longrightarrow$ it is also unique.

<u>Proof</u>. Assume that x_0, x_1 are two fixed points

$$d(x_0, x_1) = d(\varphi(x_0), \varphi(x_1)) \leq c\, d(x_0, x_1)$$

$\Rightarrow d(x_0, x_1) = 0. \Rightarrow x_0 = x_1.$

If fixed point exists, it is unique.

<u>Existence</u>: Pick arbitrary x_1,

Define inductively $x_{n+1} = \varphi(x_n)$.

$$d(x_{n+1}, x_n) \leq c\, d(x_n, x_{n-1}) \leq c^{n-1} d(x_2, x_1)$$

$m \geq n+1$

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \dots + d(x_{n+1}, x_n)$$

$$\leq (c^{m-2} + \dots + c^{n-1})\, d(x_2, x_1) \leq$$

$$\left(\sum_{k=n-1}^{\infty} c^k\right) d(x_2, x_1) = \frac{c^{n-2}}{1-c}\, d(x_2, x_1)$$

(x_n) is a Cauchy sequence.

Since X is complete,

$x_n \longrightarrow x_0$. $\quad \varphi(x_n) \longrightarrow x_0$

$d(\varphi(x), \varphi(y)) \leq c\, d(x,y) \Longrightarrow$

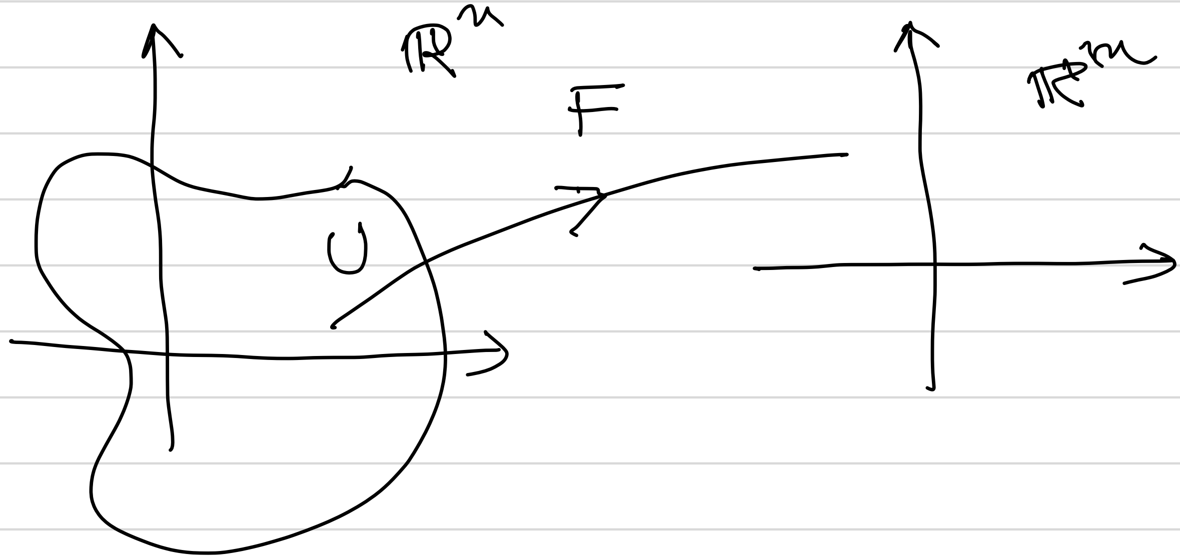
φ is continuous

$$\varphi(x_n) \longrightarrow \varphi(x_0) \Longrightarrow \varphi(x_0) = x_0$$

x_0 is a fixed point. $\boxtimes$

<u>Remarks</u>. Contraction principle is effective, gives a way to calculate the fixed point.

Differentiable maps



U is open in $\mathbb{R}^n$. $F : U \longrightarrow \mathbb{R}^m$

a function

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$$

U has natural topology given by metric on $\mathbb{R}^n$. Same with $\mathbb{R}^m$

Consider continuous map $F : U \longrightarrow \mathbb{R}^m$

$\Rightarrow F_1, \dots, F_m : U \longrightarrow \mathbb{R}$ are also continuous.

$\varepsilon > 0$, $\exists \delta > 0$ such that $\|x-y\| < \delta$

$\Rightarrow \|F(x) - F(y)\| < \varepsilon$

continuity at x.

Want to define differentiability.

Case $n=1$, $m=1$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x) - F'(x)h}{h} = 0$$

$\exists A \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|F(x+h) - F(x) - Ah|}{h} = 0$$

Generalization: $F : U \longrightarrow \mathbb{R}^m$ is differentiable at $x \in U$ if

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - A \cdot h\|}{\|h\|} = 0$$

for some $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. ← linear map $\mathbb{R}^n \to \mathbb{R}^m$

$\Rightarrow$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|h\| < \delta \Longrightarrow \|F(x+h) - F(x) - A \cdot h\| < \varepsilon \|h\|$$

① Claim. If F is differentiable at $x \in U$, F is continuous at x.

(i.e. differentiability is stronger property than continuity).

<u>Proof.</u>

$$\|F(x+h)-F(x)\| = \|F(x+h)-F(x)-$$
$$Ah+Ah\| \leq \|F(x+h)-F(x)-Ax\| + \|Ah\|$$
$$< \varepsilon\|h\| + \|Ah\|$$

<u>Recall from linear algebra:</u>

$$B(0,1) = \{ h \in \mathbb{R}^n \mid \|h\| \leq 1 \}$$

closed unit ball - compact

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$Ax = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$|x_i| < \|x\|$$

$$\Rightarrow |x_i - y_i| \leq \|x-y\|$$

$\Rightarrow$ coordinates are continuous functions on $\mathbb{R}^n$

linear functions are continuous

$$x \longmapsto \sum_{j=1}^{n} A_{ji} x_j$$

$x \longmapsto \|Ax\|$ is continuous

on $\mathbb{R}^n$

On $B(0,1)$ attains maximum.

$$\|A\| = \max_{\|x\| \leq 1} \|Ax\|$$

<u>linear map norm</u>

$$\|A+B\| = \max_{\|x\| \leq 1} \|(A+B)x\| \leq$$

$$\leq \max_{\|x\| \leq 1} \left(\|Ax\| + \|Bx\| \right) \leq$$

$$\leq \max_{\|x\|\leq 1} \|Ax\| + \max_{\|x\|\leq 1} \|Bx\| =$$

$$= \|A\| + \|B\|$$

$$\|\lambda A\| = |\lambda| \, \|A\| \quad , \quad \lambda \in \mathbb{R}$$

$x \in \mathbb{R}^n$, $x \neq 0$

$$y = \frac{x}{\|x\|} \in B(0,1)$$

$$\Rightarrow \quad \|Ay\| \leq \|A\|$$

$$\left\|A\left(\frac{x}{\|x\|}\right)\right\| \leq \|A\|$$

$$\Rightarrow \frac{1}{\|x\|} \cdot \|Ax\| \leq \|A\|$$

$$\Rightarrow \quad \|Ax\| \leq \|A\| \cdot \|x\|$$

holds also for $x = 0$.

holds for all $x \in \mathbb{R}^n$.

$\Rightarrow \quad \|A\| = 0 \Rightarrow A = 0.$

$\|.\|$ is a norm!

Back to continuity:

$$\|F(x+h) - F(x)\| < \varepsilon \cdot \|h\| + \|Ah\| \leq$$

$$\varepsilon \|h\| + \|A\| \cdot \|h\| = (\|A\| + \varepsilon) \|h\| <$$

$$(\|A\| + \varepsilon) \cdot \delta$$

$\Rightarrow$ F is continuous at x!

Claim: A is unique!

Assume the opposite.

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - Ah\|}{\|h\|} = 0$$

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - B \cdot h\|}{\|h\|} = 0$$

$\|(A-B)h\| = \|(F(x+h) - F(x) - Bh) -$

$- (F(x+h) - F(x) - Ah)\| \leq$

$\|F(x+h) - F(x) - Ah\| + \|F(x+h) - F(x) - Bh\|$

divide by $\|h\|$

$$\frac{\|(A-B)h\|}{\|h\|} \leq \frac{\|F(x+h) - F(x) - Ah\|}{\|h\|} +$$

$$+ \frac{\|F(x+h) - F(x) - Bh\|}{\|h\|}$$

as $\|h\| \to 0$ the right side tends to 0

$$\Rightarrow \lim_{h \to 0} \frac{\|(A-B)h\|}{\|h\|} = 0$$

$\Rightarrow$ For $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\|h\| < \delta \implies \|(A-B)\,h\| < \varepsilon \|h\|$$

$x \neq 0$. $\quad h = \frac{\delta}{2} \cdot \frac{x}{\|x\|} \qquad \|h\| = \frac{\delta}{2} < \delta$

$$\|(A-B)\,h\| < \varepsilon \cdot \frac{\delta}{2}$$

$$\Rightarrow \|(A-B)\,x\| = \frac{2}{\delta} \cdot \|x\| \cdot \|(A-B)\,h\| < \varepsilon \|x\|$$

$$\Rightarrow \left\|(A-B)\left(\frac{x}{\|x\|}\right)\right\| < \varepsilon$$

$$\Rightarrow \|A-B\| < \varepsilon \implies \|A-B\| = 0$$

$$A - B = 0 \implies \boxed{A = B} \ !$$

A is completely determined by F !

Def: $\boxed{\boxed{F'(x) = A}}$

This is the <u>differential</u> (or <u>derivative</u>) of F at x,

(not to be confused with partial derivative!)