We say that X is a <u>topological space</u> with the topology $\mathcal{U}$ if $\mathcal{U}$ is a family of subsets of X satisfying

(T1) ϕ, X are in $\mathcal{U}$;

(T2) if $(U_i, i \in I)$ is a family of elements of $\mathcal{U}$ then $\bigcup_{i \in I} U_i \in \mathcal{U}$;

(T3) if $U_1, U_2, \ldots, U_n$ is a finite family of elements of $\mathcal{U}$ then $U_1 \cap \ldots \cap U_n$ is in $\mathcal{U}$.

Closed sets

X topological space
$\mathcal{U}$ - family of all open sets in X

A set $Z \subset X$ is closed if $X - Z$ is open.

① $\emptyset$ and X are also closed sets.
② $(Z_i, i \in I)$ family of closed

sets. Then $U_i = X \setminus Z_i$ are open. Hence

$U = \bigcup_{i \in I} U_i$ is open.

$$X \setminus U = X \setminus \left(\bigcup_{i \in I} U_i \right) =$$

$$= \bigcap_{i \in I} (X \setminus U_i) = \bigcap_{i \in I} Z_i$$

Hence, <u>the intersection of any family of closed sets is closed.</u>

③ $Z_1, Z_2, \dots, Z_n$ finite family of closed sets. Then $U_i = X \setminus Z_i$ are open. Hence, $U_1 \cap U_2 \cap \dots \cap U_n$ is open

$$X \setminus (U_1 \cap \dots \cap U_n) =$$

$$= (X \setminus U_1) \cup (X \setminus U_2) \cup \dots \cup (X \setminus U_n)$$

$$= Z_1 \cup Z_2 \cup \dots \cup Z_n$$

$\Rightarrow Z_1 \cup Z_2 \cup \dots \cup Z_n$ is closed.

Hence, finite union of closed sets is closed.

Example. In discrete space X all subsets of X are also closed.

The family of all open subsets of X is called the topology on X.

Any metric space is a topological space. Converse is false.

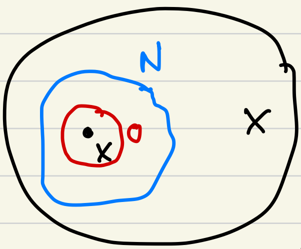
<u>Example</u>. X a set, $\mathcal{U} = \{\phi, X\}$ is a topology on X. It is called the <u>chaos topology</u>.

Assume that X is a metric space. $x_0 \in X$. Then $B(x_0, \varepsilon)$ contains x_0. Hence it is a nonempty open set $\Longrightarrow B(x_0, \varepsilon) = X$ for any $\varepsilon > 0$.

If $x_1 \in X$ is a point different from x_0, $x_1 \in B(x_0, \varepsilon) \Longrightarrow$
$d(x_0, x_1) < \varepsilon$ for any $\varepsilon > 0$
$\Longrightarrow d(x_0, x_1) = 0 \Longrightarrow x_0 = x_1$
and we have a contradiction.
So, if X has more than two points it cannot be a metric space.
(This is a stupid example.

We shall see more interesting examples later :)

If X is a topological space and $x \in X$ a point, we say that a subset $N \subset X$ is a <u>neighborhood</u> of x if there exists an open set O such that $x \in O \subset N$.

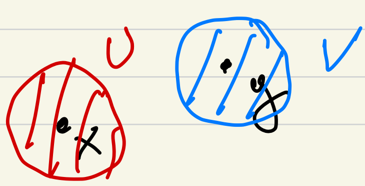


An open set of X containing x is an (open) neighborhood of x.

Ex: If X is a topological space with chaos topology & x a point in X, the only neighborhood of x is X.

Hence if X contains two points x, y, $x \neq y$ their neighborhoods intersect.

Two points x, y in a topological space X are separated if there exists neighborhoods U of x and V of y such that $U \cap V = \emptyset$.



A topological

space X is hausdorff if any two different points are separated.

Example: ① A topological space with chaos topology and more than one point is not hausdorff.

② Metric spaces are hausdorff.

Let x, y be two points in X. Assume that $x \neq y$.

Then $\delta = d(x,y) > 0$.

Let $z \in B(x, \frac{\delta}{2}) \cap B(y, \frac{\delta}{2})$.

Then

$$d(x,y) \leq d(x,z) + d(z,y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

Contradiction! $B(x, \frac{\delta}{2}) \cap B(y, \frac{\delta}{2}) = \emptyset$

X topological space

$S \subset X$ subset

$\mathcal{F}$ family of all open subsets of S – nonempty (contains empty set)

Union of all elements of $\mathcal{F}$ is open and contained in $\mathcal{F}$ – largest element of $\mathcal{F}$

Fact. There exists the largest open subset of S – <u>interior</u> of S - $\mathrm{int}(S)$

$\mathrm{int}(S) \subset S$.

$\mathcal{C}$ - family of all closed sets in X containing S

- nonempty (contains X)

intersection of all elements in $\mathcal{C}$ is closed and contains S $\Longrightarrow$ it is in $\mathcal{C}$

$\Rightarrow$ the smallest element of $\mathcal{C}$

There exists the smallest closed set containing S - the <u>closure</u> of S - $\overline{S}$.

$$\operatorname{int}(S) \subset S \subset \overline{S}$$

$$\partial S = \bar{S} \setminus \operatorname{int}(S) = \underbrace{\bar{S} \cap (X \setminus \operatorname{int}(S))}_{\text{closed}}$$

closed set

- <u>boundary</u> of S.

<u>Examples</u>

① $B(x_0, \varepsilon)$ is open

$\bar{B}(x_0, \varepsilon) = \{x \in X \mid d(x_0, x) \leq \varepsilon\}$

is closed

<u>$\operatorname{int}(B(x_0,\varepsilon))$</u> $= B(x_0, \varepsilon)$

$\overline{B(x_0, \varepsilon)} = \bar{B}(x_0, \varepsilon)$

$\partial B(x_0, \varepsilon) = \{x \in X \mid d(x_0, x) = \varepsilon\}$

② $\mathbb{Q} \subset \mathbb{R}$. $\operatorname{int}(\mathbb{Q}) = \emptyset$

$\bar{\mathbb{Q}} = \mathbb{R}$. $\partial \mathbb{Q} = \mathbb{R}$!