ℓ^2 is complete.

(u_n) a Cauchy sequence in ℓ^2

$u_n = (u_{n,1}, \dots, u_{n,p}, \dots)$

$\varepsilon > 0 \ \exists N$ such that

$d(u_n, u_m) < \varepsilon$ for $n, m \geq N$.

$$\|u_n - u_m\|^2 = \sum_{i=1}^{\infty} |u_{n,i} - u_{m,i}|^2 < \varepsilon^2$$

$\Rightarrow (u_{n,i})$ satisfy

$n, m \geq N \Longrightarrow |u_{n,i} - u_{m,i}| < \varepsilon$.

$\Rightarrow (u_{n,i})$ is a Cauchy sequence in $\mathbb{C}$

$u_{n,i} \longrightarrow v_i$ in $\mathbb{C}$.

$\Rightarrow$
$$\sum_{i=1}^{M} |u_{n,i} - u_{m,i}|^2 < \varepsilon^2$$

take the limit $m \to \infty$

$$\sum_{i=1}^{M} |u_{n,i} - v_i|^2 \leq \varepsilon^2$$

$$\Rightarrow \lim_{M \to \infty} \sum_{i=1}^{M} |u_{n,i} - v_i|^2 \leq \varepsilon^2$$

$$\sum_{i=1}^{\infty} |u_{n,i} - v_i|^2 \leq \varepsilon^2$$

$$\Rightarrow u_n - v \in \ell^2$$

$$\Rightarrow v = (v - u_n) + u_n \in \ell^2$$

$$\|u_n - v\| \leq \varepsilon \quad \text{for } n \geq N$$

$$\Rightarrow u_n \longrightarrow v \text{ in } \ell^2$$

ℓ^2 is complete.

ℓ^2 is a Hilbert space.

There exist infinite dimensional Hilbert spaces.

ℓ^2 , $e_i = (0, \dots, 0, 1, 0, \dots)$

i^{th} place

$(e_i ; 1 \le i < \infty)$ is a linearly independent family.

- linear span is

$\mathbb{C}^\infty = \{ (c_1, c_2, \dots) \mid c_i \in \mathbb{C}$ - finitely many of them nonzero $\}$

- a dense subspace of ℓ^2
- not complete

Example

$u = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

$u \in \ell^2$

$u_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots)$

$u_n \to u$ in ℓ^2

$u_n \in \mathbb{C}^\infty$, but $u \notin \mathbb{C}^\infty$.

$$(e_i | e_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$u = (c_1, c_2, \ldots)$

$$u = \sum_{i=1}^{\infty} c_i e_i$$

- partial sums are

$$u_n = \sum_{i=1}^{n} c_i e_i = (c_1, c_2, \ldots, c_n, 0, \ldots)$$

$u_n \to u$ in ℓ^2

$$\|u_n - u\|^2 = \sum_{i=n+1}^{\infty} |c_i|^2 \to 0$$

as $n \to \infty$.

$$u = \sum_{i=1}^{\infty} c_i e_i$$

$(e_i; i \in \mathbb{N})$ is a <u>Hilbert space</u> orthonormal basis of ℓ^2.

- Not an orthonormal basis in sense of linear algebra!
- Maximal orthonormal set

$(u|e_i) = 0$ for all i

$u = (c_1, c_2, \dots)$

$(u|e_i) = c_i \Rightarrow c_i = 0 \Rightarrow u = 0.$

V inner product space

$(u|v)$ inner product

$(u,v) \longrightarrow (u|v)$ is a cont function on $V\times V$

$$|(u|v)-(u_0|v_0)| = |(u|v) - (u_0|v) +$$
$$+ (u_0|v) - (u_0|v_0)| = |(u-u_0|v) +$$
$$+ (u_0|v-v_0)| \le |(u-v_0|v)| + |(u_0|v-v_0)| \le$$
$$\le \|u-u_0\|\cdot\|v\| + \|u_0\|\cdot\|v-v_0\|.$$

inner product is continuous on $V\times V$.

$(e_i;\, i\in\mathbb{N})$ orthonormal system in V

Assume that

$$v\in V \qquad v = \sum_{i=1}^{\infty} c_i e_i$$

$$(v|v) = \lim_{n\to\infty} (v_n|v)$$

$$v_n = \sum_{i=1}^{n} c_i e_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} c_i (e_i|v) =$$

$$= \sum_{i=1}^{\infty} c_i (e_i|v)$$

$$(e_i|v) = \lim_{m\to\infty} \sum_{j=1}^{m} (e_i|c_j e_j) =$$

$$= \lim_{m\to\infty} \sum_{j=1}^{m} \overline{c_j} (e_i|e_j) = \overline{c_i}$$

$\Rightarrow$

$$(v|v) = \sum_{i=1}^{\infty} |c_i|^2$$

Bessel equality

(Hilbert space) orthonormal basis

Bessel equality

$$\sum_{i=1}^{\infty} |c_i|^2 = \|v\|^2$$

$$\varphi : V \ni v \longmapsto (c_1, c_2, \ldots, c_n, \ldots) \in \ell^2$$

φ is a linear map

injective

Can identify V with $\varphi(V)$

$\varphi(V)$ is dense in ℓ^2

(it contains $U = \mathbb{C}^\infty$)

$(\,|\,)$ on V agrees with inner product in ℓ^2

$$(u|v) = \sum_{i=1}^{\infty} u_i \overline{v_i}$$

Any inner product space with (Hilbert space) orthonormal basis is a subspace of a Hilbert space (it has a Hilbert space completion).

<u>Example</u>

$\mathcal{C}_{per}(\mathbb{R})$ - periodic continuous functions on $\mathbb{R}$ with period 2π.

$f, g \in \mathcal{C}_{per}(\mathbb{R})$

$$(f|g) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$$

$e_n(x) = e^{inx}$ $\quad (e_n;\ n \in \mathbb{Z})$

is an orthonormal system

$(e_0, e_1, e_{-1}, e_2, e_{-2}, \dots)$

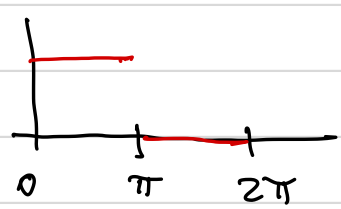
order differently.

$\mathcal{C}_{per}(\mathbb{R}) \longrightarrow \ell^2$

$\mathcal{C}_{per}(\mathbb{R})$ has a Hilbert space completion!

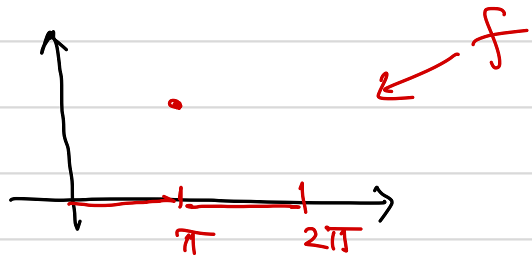
Question. Are its elements functions?

- replace continuity of f by Riemann integrability of $|f|^2$ on $[0,2\pi]$
- gives "bigger" space



← is in the space

Problem ①



$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx = 0$$

$$\| \quad \|f\|^2$$

but $f \neq 0$.

One has to identify functions f and g such that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)-g(x)|^2\,dx = 0$$

- classes of functions

$f \sim g$, if f and g are continuous $\Rightarrow f = g$!

② still not complete — one has to introduce Lebesgue integral!