Cauchy sequences and completeness

(X,d) metric space, d – metric on X.

$\mathbb{N} \ni n \longmapsto x_n \in X$

a function from $\mathbb{N}$ to X – a sequence in X.

A sequence (x_n) is convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$; i.e., for any $\varepsilon > 0$ there exists n_0 such that $n \geq n_0 \Longrightarrow d(x_n, x) < \varepsilon$.

In homework, you proved that a sequence has at most one limit x.

$\lim x_n = x$.

<u>Cauchy sequence</u> is a sequence (x_n) such that for any $\varepsilon > 0$ there exists n_0 such that

$$d(x_n, x_m) < \varepsilon \quad \text{for } n, m \geq n_0.$$

<u>Theorem</u>. A convergent sequence is a Cauchy sequence.

<u>Proof</u>: Let (x_n) be a convergent sequence and $x = \lim x_n$.

Let $\varepsilon > 0$. There exists n_0 such that $d(x_n, x) < \varepsilon/2$ for $n \geq n_0$.

Hence

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) <$$

$$\varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $n, m \geq n_0$.

Hence (x_n) is a Cauchy sequence, $\blacksquare$

A metric space (X, d) is <u>complete</u> if every Cauchy sequence (x_n) in X converges.

<u>Examples</u>:

1. $\mathbb{R}$ with $d(x,y) = |x-y|$ is complete.
2. $\mathbb{Q}$ with $d(x,y) = |x-y|$ is not complete.

A way to construct $\mathbb{R}$ from $\mathbb{Q}$

is to add "missing" limits of Cauchy sequences.

This is a consequence of a more general result:

For any metric space (X,d) there exists a metric space $(\hat{X},\hat{d})$ such that

(i) $X \subset \hat{X}$;

(ii) X is dense in $\hat{X}$;

(iii) $d(x,y) = \hat{d}(x,y)$, $x,y \in X$.

(iv) $\hat{X}$ is complete.

$\hat{X}$ is called the <u>completion</u> of X.

Let V be a inner product space with inner product $(\cdot|\cdot)$. Then $\|u\|^2 = (u|u)$ is a norm on V and $d(u,v) = \|u-v\|$ is a metric on V.

V is a <u>Hilbert space</u> if (V,d) is complete.

<u>Facts</u>:

① Finite dimensional inner product spaces are complete - Hilbert spaces.

② $\mathcal{C}([0,1])$ $\quad (f|g) = \int_0^1 f(x)\overline{g(x)}\,dx$

is <u>not</u> complete - not a

Hilbert space.

① Let $(e_1, \dots, e_n)$ be an ortho-normal basis of V.

Then $u = \sum_{i=1}^{n} \alpha_i e_i$, $v = \sum_{i=1}^{n} \beta_i e_i$

$$(u|v) = \left(\sum_{i=1}^{n} \alpha_i e_i \,\Big|\, \sum_{j=1}^{n} \beta_j e_j\right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} (e_i | e_j) =$$

$$= \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

$$\|u\|^2 = \sum_{i=1}^{n} |\alpha_i|^2$$

$$d(u,v) = \sum_{i=1}^{n} |\alpha_i - \beta_i|^2.$$

(u_n) a Cauchy sequence in V, $u_n = \sum_{i=1}^{n} \alpha_{n,i} e_i$

$$\|u_n - u_m\|^2 = \sum_{i=1}^{n} |\alpha_{n,i} - \alpha_{m,i}|^2$$

$d(u_n, u_m) < \varepsilon$

$\Rightarrow \quad |\alpha_{n,i} - \alpha_{m,i}| < \varepsilon$

(u_n) Cauchy sequence in V

$\Rightarrow \quad (\alpha_{n,i})$ Cauchy sequence in $\mathbb{C}$ for $1 \leq i \leq n$.

<u>Fact</u> : $\mathbb{C}$ is complete

$|z|^2 = x^2 + y^2 \qquad z = x + iy$

(z_n) Cauchy sequence of complex numbers

$|z_n - z_m| < \varepsilon$

$\Rightarrow \quad |x_n - x_m|^2 + |y_n - y_m|^2 < \varepsilon^2$

$\Rightarrow \quad |x_n - x_m| < \varepsilon \;\&\; |y_n - y_m| < \varepsilon$

(x_n) is a Cauchy sequence of real numbers

(y_n) is a Cauchy sequence of real numbers

$x_n \to x$ ($\mathbb{R}$ is complete)

$y_n \to y$

$z_n \to z = x + iy$!

$\mathbb{C}$ is complete. $\boxtimes$

Hence $\alpha_{n,i} \to \alpha_i \in \mathbb{C}$

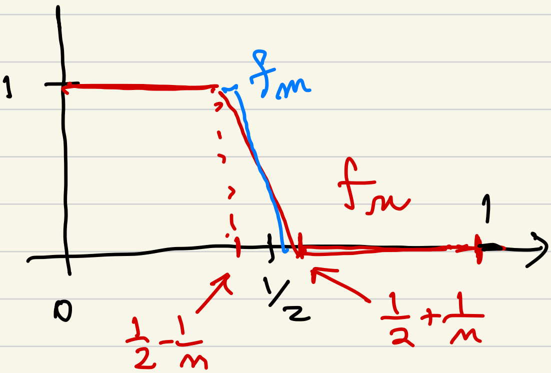
$$u = \sum_{i=1}^{n} \alpha_i e_i$$

$\Rightarrow d(u_n, u) \to 0$ as $n \to \infty$. $\lim u_n = u$.

(u_n) is convergent.

V is complete.

$\mathcal{C}([0,1])$



$n < m$

$f_n - f_m$ is 0 $\quad 0 \leq x \leq \frac{1}{2} - \frac{1}{n}$

$\qquad\qquad\qquad\quad \frac{1}{2} + \frac{1}{n} \leq x \leq 1$

$$-1 \leq (f_n - f_m)(x) \leq 1$$

$$\|f_n - f_m\|^2 = \int_0^1 |f_n(x) - f_m(x)|^2 \, dx \leq$$

$$\leq \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} 1 \cdot dx = \frac{2}{n} \to 0$$

(f_n) is a Cauchy sequence.

Assume that $f_n \longrightarrow f$ in $\mathscr{C}([0,1])$ for $f \in \mathscr{C}([0,1])$. Then

$$\|f_n - f\| \longrightarrow 0$$

$$f_n(x) \longrightarrow g(x) \text{ on } [0,1]$$

$$g(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & \frac{1}{2} x \le 1 \end{cases}.$$

$$\int_0^1 |f_n(x) - g(x)|^2 dx = \int_0^{1/2} |f_n(x) - g(x)|^2 dx$$

$$+ \int_{1/2}^1 |f_n(x) - g(x)|^2 dx \le \frac{1}{n} + \frac{1}{n} \longrightarrow 0$$

$$\int_0^1 |f(x) - g(x)|^2 dx = \int_0^{1/2} |f(x) - g(x)|^2 dx$$

$$\int_{1/2}^1 |f(x) - g(x)|^2 dx$$

$$+ \int_{1/2}^{1} |f(x)|^2 dx .$$

$$\int_0^{1/2} |f_n(x) - f(x)|^2 dx \le \int_0^1 |f_n(x) - f(x)|^2 dx$$

$$\int_{1/2}^{1} |f_n(x) - f(x)|^2 dx \le \underbrace{\int_0^1 |f_n(x) - f(x)|^2 dx}_{\to 0}$$

$$\int_0^{1/2} |f_n(x) - g(x)|^2 dx = \int_0^{1/2} |f_n(x) - 1|^2 dx$$

$$\to 0 \text{ as } n \to \infty$$

$$\int_{1/2}^{1} |f_n(x) - g(x)| dx = \int_{1/2}^{1} |f_n(x)|^2 dx$$

$$\left(\int_0^{1/2} |f(x) - 1|^2 dx \right)^{1/2} \le \underbrace{\left(\int_0^{1/2} |f(x) - f_n(x)|^2 dx \right)^{1/2}}_{\to 0} +$$

$$+ \underbrace{\left(\int_0^{1/2} |f_n(x) - 1|^2 dx \right)^{1/2}}_{\to 0}$$

$$\Rightarrow \int_0^{1/2} |f(x) - 1|^2 dx = 0$$

$$\left(\int_{1/2}^{1} |f(x)|^2 dx\right)^{2} \leq$$

$$\underbrace{\left(\int_{1/2}^{1} |f(x)-f_n(x)|^2\right)^{1/2}}_{\to 0} + \underbrace{\left(\int_{1/2}^{1} |f_n(x)|^2 dx\right)^{1/2}}_{\to 0}$$

$$\Rightarrow |f(x)-1| = 0 \quad \text{for } x \in [0, 1/2]$$

$$|f(x)| = 0 \quad \text{for } x \in [1/2, 1]$$

$\Rightarrow f(x) = 1$ for $x \in [0, 1/2]$, $f(x) = 0$ for $x \in [1/2, 1]$ what is impossible.

So, f cannot be continuous. (f_n) does not converge in $\mathcal{C}([0,1])$.

$$\ell^2 = \left\{ (c_1, c_2, \dots) \mid c_i \in \mathbb{C}, \sum_{i=1}^{\infty} |c_i|^2 < +\infty \right\}$$

$u, v \in \ell^2$

$$u + v = (u_1 + v_1, u_2 + v_2, \dots)$$

$$\left(\sum_{i=1}^{N} |u_i + v_i|^2 \right)^{1/2} \le \left(\sum_{i=1}^{N} |u_i|^2 \right)^{1/2} +$$

$$\left(\sum_{i=1}^{N} |v_i|^2 \right)^{1/2} \le \left(\sum_{i=1}^{\infty} |u_i|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |v_i|^2 \right)^{1/2}$$

for all N. $N \longmapsto \sum_{i=1}^{N} |u_i + v_i|^2$ is increasing sequence and bounded

$\Rightarrow$ convergent!

$$\Rightarrow \sum_{i=1}^{\infty} |u_i + v_i|^2 < +\infty \Longrightarrow u + v \in \ell^2$$

$\alpha \in \mathbb{C}$, $\alpha \cdot u = (\alpha \cdot u_1, \alpha \cdot u_2, \dots)$

$$\sum_{i=1}^{\infty} |\alpha u_i|^2 = |\alpha|^2 \cdot \sum_{i=1}^{\infty} |u_i|^2 < +\infty$$

$a \cdot u \in \ell^2$

ℓ^2 is a vector space over $\mathbb{C}$.

$u, v \in \ell^2$

$$(u|v) = \sum_{i=1}^{\infty} u_i \overline{v_i}$$

The series on right converges absolutely

$$\left(\sum_{i=1}^{N} |u_i| \cdot |v_i| \right)^2 \leq \sum_{i=1}^{N} |u_i|^2 \cdot \sum_{i=1}^{N} |v_i|^2$$

by Cauchy-Schwartz for $\mathbb{C}^n$.

$$\leq \left(\sum_{i=1}^{\infty} |u_i|^2 \right) \cdot \left(\sum_{i=1}^{\infty} |v_i|^2 \right)$$

$N \longmapsto \sum_{i=1}^{N} |u_i| \cdot |v_i|$ is increasing and bounded $\Rightarrow$ convergent.

$\sum_{i=1}^{\infty} u_i \overline{v_i}$ is absolutely convergent.

$(u+v|w) =$

$$\sum_{i=1}^{\infty}(u_i+v_i)\overline{w_i} = \sum_{i=1}^{\infty} u_i\overline{w_i} + \sum_{i=1}^{\infty} v_i\overline{w_i} =$$

$= (u|w) + (v|w)$

$$(\alpha u|w) = \sum_{i=1}^{\infty} \alpha u_i \overline{w_i} = \alpha \sum_{i=1}^{\infty} u_i \overline{w_i} =$$

$= \alpha(u|w)$

$$\overline{(u|v)} = \overline{\sum_{i=1}^{\infty} u_i \overline{v_i}} = \sum_{i=1}^{\infty} \overline{u_i}\,\overline{\overline{v_i}} =$$

$$= \sum_{i=1}^{\infty} \overline{u_i} v_i = (v|u)$$

$$(u|u) = \sum_{i=1}^{\infty} |u_i|^2 \geq 0$$

if $= 0 \Rightarrow |u_i|^2 = 0$ for all i

$\Rightarrow u_i = 0 \Rightarrow u = 0$.

$(\cdot|\cdot)$ is an inner product. ℓ^2 is an inner product space.