$\mathcal{C}_{per}(\mathbb{R})$

$$(f|g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$$

inner product

$$e_n(x) = e^{inx} \qquad n \in \mathbb{Z}$$

$$(e_n|e_m) = 0 \text{ if } n \neq m$$

$$(e_n|e_n) = 1$$

$$\|f\| = (f|f)^{1/2}$$

$$|||f||| = \max_{x\in[0,2\pi]} |f(x)|$$

another norm

$$\|f\|^2 = (f|f) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx \leq$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |||f|||^2 dx = |||f|||^2$$

$$\|f\| \leq |||f|||$$

These norms define two different metric on $\mathcal{C}_{per}(\mathbb{R})$ (and - different topologies).

$$B(f,\varepsilon) = \{g \mid |||g-f||| < \varepsilon\}$$

$$\underline{B}(f,\varepsilon) = \{g \mid \|g-f\| < \varepsilon\}$$

By the above inequality

$$B(f,\varepsilon) \subset \underline{B}(f,\varepsilon)$$

for any f and ε.

If U is open set for topology given by $\|.\|$ any $f \in U$ has a neighborhood $\underline{B}(f,\varepsilon) \subset U$.

$\Rightarrow B(f,\varepsilon) \subset U \Rightarrow U$ is open

in topology defined by $|||.|||$.
Topology defined by $|||.|||$ has more open sets - it is <u>finer</u> than the topology defined by $||.||$.

$e_{-n}, e_{-n+1}, \dots, e_{n-1}, e_n$ span a vector subspace U of $\mathcal{C}_{per}(\mathbb{R})$ consisting of trig polynomials $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$.

As we remarked, for a function $f \in \mathcal{C}_{per}(\mathbb{R})$

$$\left\| f - \sum_{k=-n}^{n} (f|e_k)\, e_k \right\|$$

$$\leq \| f - P \|$$

for any $P \in U$.

Let $\varepsilon > 0$. By Stone-Weierstrass, there exists a trig polynomial $P = \sum_{k=-n}^{n} a_k e_k$ such that

$$|||f - P||| < \varepsilon.$$

This implies that

$$\|f - P\| < \varepsilon.$$

Therefore, for that n

$$\|f - \sum_{k=-N}^{N} (f|e_k) e_k\| \leq \|f - P\| < \varepsilon.$$

for any $N \geq n$ (since P is in the subspace spanned by $(e_k; -N \leq k \leq N)$.

Therefore, we have the sequence

of partial sums

$$S_N : N \longmapsto \sum_{k=-N}^{N} (f|e_k)\, e_k$$

such that continuous functions (S_N) converge to f, i.e.,

$$d(S_N, f) < \varepsilon \quad \text{for } N \geq n.$$

($\lim S_N = f$.

this is the metric defined by the norm $\|.\|$ (and not the "uniform" norm $|||.|||$ – this is not uniform convergence).

We have, by Pythagora's theorem,

$$\|f\|^2 = \left\|f - \sum_{k=-N}^{N} (f|e_k)\, e_k\right\|^2 +$$

$$+ \left\| \sum_{k=-N}^{N} (f|e_k) e_k \right\|^2 =$$

$$= \left\| f - \sum_{k=-N}^{N} (f|e_k) e_k \right\|^2 + \sum_{k=-N}^{N} |(f|e_k)|^2$$

$\sum_{k=-N}^{N} |(f|e_k)|^2$ has positive terms, it is increasing as $N \to \infty$ and bounded by Bessel inequality.

$\Rightarrow$ converges!

By taking the limit, we get

$$\boxed{\|f\|^2 = \sum_{k=-\infty}^{\infty} |(f|e_k)|^2 .}$$

This is the Bessel equality.

Assume that $f \in \mathcal{C}_{per}(\mathbb{R})$ has continuous derivative. Then

$$(f' | e_n) = \frac{1}{2\pi} \int_0^{2\pi} f'(x)\, e^{-inx}\, dx =$$

$$= \underbrace{\frac{1}{2\pi} \left[f(x)\, e^{-inx} \right]_0^{2\pi}}_{=0} + \frac{in}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

$$= in\, (f | e_n)$$

by integration by parts.

$$|(f' | e_n)| = n\, |(f | e_n)|.$$

If f is twice continuously differentiable

$$|(f'' | e_n)| = n\, |(f' | e_n)| =$$

$$n^2\, |(f | e_n)|.$$

By Bessel inequality,

$$\sum_{k=-n}^{n} |(f|e_k)|^2 \leq \|f\|^2$$

$$\Longrightarrow \quad |(f|e_n)| \leq \|f\|$$

for any $n \in \mathbb{Z}$ (or by direct estimate).

$$\Longrightarrow \quad |(f''|e_n)| \leq \|f''\|$$

for all $n \in \mathbb{Z}$.

$$\Longrightarrow \quad n^2 |(f|e_n)| \leq \|f''\|.$$

$$\boxed{|(f|e_n)| \leq \frac{\|f''\|}{n^2}}$$

for $n \neq 0$.

Since $\sum_{k=1}^{\infty} \frac{1}{n^2}$ is convergent

by integral test,

$$\sum_{n=-\infty}^{\infty} |(f|e_n)|$$

is convergent

$$F(x) = \sum_{n=-\infty}^{\infty} (f|e_n)\, e_n$$

is absolutely convergent,

$$\left| F(x) - \sum_{n=-N}^{N} (f|e_n)\, e^{inx} \right| =$$

$$= \left| \sum_{n=N+1}^{\infty} (f|e_{-n}) e^{-inx} - \sum_{n=N+1}^{\infty} (f|e_n) e^{inx} \right| \leq$$

$$\leq \sum_{|n| \geq N+1}^{\infty} |(f|e_n)| < \varepsilon$$

for large N.

Therefore, the above

convergence is <u>uniform</u>.

This implies that F is a continuous function.

Uniform convergence (with respect to norm $|||\cdot|||$) implies the convergence with respect to norm $\|\cdot\|$. Hence

$$F = \sum_{n=-\infty}^{\infty} (f|e_n)\, e_n = f \,!$$

Therefore we proved that the Fourier series of f converge to f uniformly if f is twice differentiable!