Complex Cauchy-Schwartz inequality

V inner product space over $\mathbb{C}$

$$|(u|v)| \leq \|u\| \cdot \|v\|$$

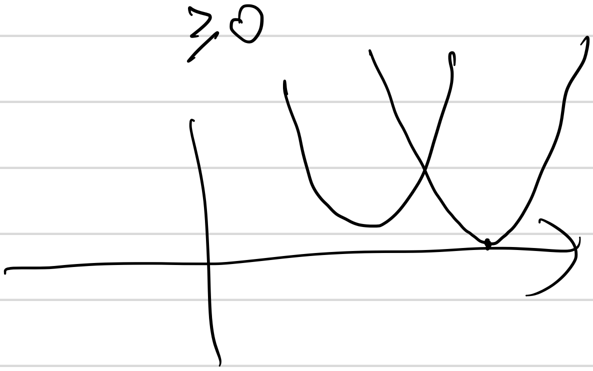
$$\mathbb{R} \ni t \longmapsto (u+tv\,|\,u+tv) \geq 0$$

$$\shortparallel$$

$$(u|u) + t(v|u) + t(u|v) + t^2(v|v)$$

$$\shortparallel$$

$$\underbrace{(u|u)}_{\geq 0} + 2\mathrm{Re}(u|v)\cdot t + t^2\underbrace{(v|v)}_{\geq 0}$$



the parabola doesn't go below x-axes

$$ax^2 + bx + c \geq 0$$

$$a = (v|v),\ c = (u|u) \quad b = 2\mathrm{Re}(u|v)$$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac \leq 0$ (if there are no two real solutions)

$$\cancel{4}\left(\mathrm{Re}(u|v)\right)^2 - \cancel{4}\|u\|^2 \cdot \|v\|^2 \leq 0$$

$$|\mathrm{Re}(u|v)| \leq \|u\| \cdot \|v\|$$

$$(u|v) = |(u|v)| \cdot e^{i\varphi}$$

$$e^{-i\varphi}(u|v) = |(u|v)|$$

$$(e^{-i\varphi}u|v) = |(u|v)|$$

$$\overset{\|}{|\mathrm{Re}(e^{-i\varphi}u|v)|}$$

$$|(u|v)| = |\mathrm{Re}(e^{-i\varphi}u|v)| \leq$$

$$\leq \|e^{-i\varphi}u\| \cdot \|v\| = \|u\| \cdot \|v\| .\quad \text{C-S !}$$

$$\|u+v\|^2 = (u+v|u+v) =$$
$$= (u|u) + (u|v) + (v|u) + (v|v) =$$
$$= \|u\|^2 + 2\,\mathrm{Re}(u|v) + \|v\|^2 \leq$$
$$\leq \|u\|^2 + 2|(u|v)| + \|v\|^2 \leq$$
$$\leq \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2 =$$
$$= (\|u\| + \|v\|)^2$$
$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|$$

$\|u\| \geq 0$

$\|\lambda u\| = |\lambda| \cdot \|u\|$

– triangle equality.

Hence $\|.\|$ is a norm on V.

V complex vector space with inner product $(\mid)$.

$e_1, \dots, e_n$ orthonormal set of vectors

$$(e_i|e_j) = 0 \quad i \neq j \quad (e_i|e_i) = 1.$$

U spanned by $e_1, \dots, e_n$

$u \in V$

$$u = a + b$$

$$a = \sum_{i=1}^{n} (u|e_i) e_i$$

$$b = u - a = u - \sum_{i=1}^{n} (u|e_i) e_i$$

$$(b|e_i) = (u|e_i) - \sum_{j=1}^{n} (u|e_j)(e_j|e_i) = 0$$

$b \perp U$, $b \perp a$.

$\|u\|^2 = \|a\|^2 + \|b\|^2$. - Pythagora

$$d = \sum_{i=1}^{n} c_i e_i$$

$$\|u-d\|^2 = \|(u-a)-(d-a)\|^2$$

$$= \|u-a\|^2 + \|a-d\|^2 =$$

$$= \|u-a\|^2 + \left\|\sum_{i=1}^{n}((u|e_i)-c_i)e_i\right\|^2 =$$

$$= \|u-a\|^2 + \sum_{i=1}^{n} |(u|e_i)-c_i|^2.$$

The distance between u and $d \in U$ is minimal for $d=a$, This implies that a is the closest point in U to u.