$\mathcal{C}_{per}(\mathbb{R})$

$$(f|g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$$

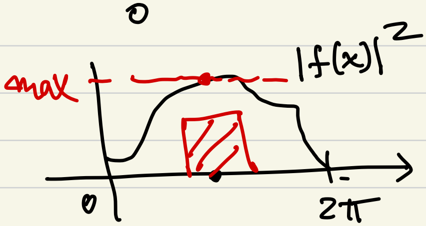
inner product on $\mathcal{C}_{per}(\mathbb{R})$.

- linear in first variable
- antilinear in second variable

$$(f|f) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx \geq 0$$

If $(f|f) = 0$

$$\int_0^{2\pi} |f(x)|^2 dx = 0$$



$\Longrightarrow \max |f(x)|^2 = 0 \Longrightarrow$

$|f(x)| = 0 \Rightarrow f(x) = 0$ for all x.

$f = 0$.

$$e_n = e^{inx}, \quad n \in \mathbb{Z}$$

$$(e_n | e_m) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{i(n-m)x}}{i(n-m)} \right]_0^{2\pi} = 0$$

if $n \neq m$

$$(e_n | e_n) = \frac{1}{2\pi} \int_0^{2\pi} dx = 1.$$

$(e_n ; n \in \mathbb{Z})$ is an orthonormal family of vectors

$\Rightarrow$ they are linearly independent

$\Rightarrow$ $\mathcal{C}_{per}(\mathbb{R})$ is infinite dimensional

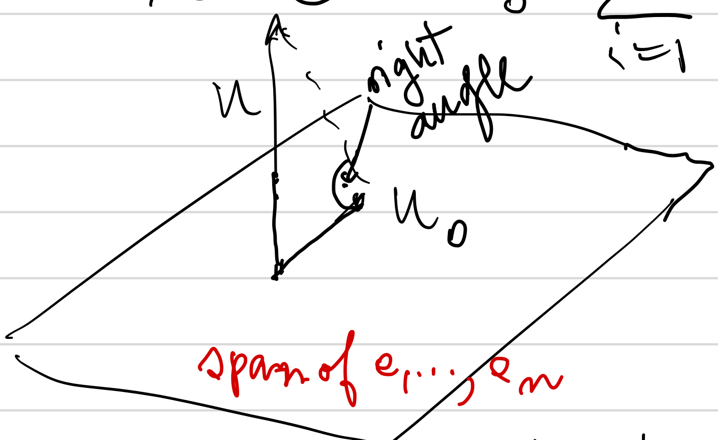
Orthonormal systems

U a complex vector space with inner product $(\mid)$

$(e_1, e_2, \dots, e_n)$ a set of orthonormal vectors

$$(e_i \mid e_j) = \delta_{ij}$$

$$u \in U \qquad u_0 = \sum_{i=1}^{n} c_i e_i \qquad c_i = (u \mid e_i)$$



$$(u - u_0 \mid e_i) = 0$$

for all i

$u - u_0$ is orthogonal to any $v \in \operatorname{span}\{e_1, \dots, e_n\}$

Let $v \in \operatorname{span}\{e_1, \dots, e_n\}$

Then, $v - u_0 \in \operatorname{span}\{e_1, \dots, e_n\}$

$$u - v = u - u_0 + (u_0 - v)$$
$$\perp$$

Pytagora $\Longrightarrow \|u-v\|^2 = \|u-u_0\|^2 + \|u_0 - v\|^2$

The function $V \ni v \longmapsto \|u-v\|$ attains its minimum at u_0. It is $\|u-u_0\|$. <u>The best approximation</u> of u in the span of $\{e_1, \ldots, e_n\}$.

$$\|u_0\|^2 = \sum_{i=1}^{n} |c_i|^2$$

$$\|u\| = \sum_{i=1}^{n} |c_i|^2 + \|u-u_0\|^2$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} |c_i|^2 \le \|u\|^2}$$

<u>Bessel inequality</u>.