Complex version : $K$ compact space

$\mathcal{C}(K)$ - complex continuous functions on $K$

$$d(f,g) = \max_{x \in K} |f(x) - g(x)|$$

$A \subset \mathcal{C}(K)$ complex subalgebra

(a) $x_1, x_2 \in K$ $\exists f \in A$ $f(x_1) \neq f(x_2)$;

(b) $x \in K$ $\exists f \in A$ $f(x) \neq 0$

(c) $f \in A \Longrightarrow \bar{f} \in A$ selfadjoint

Proof. Def $\mathcal{C}_{\mathbb{R}}(K)$ - real valued functions

$A_{\mathbb{R}} = A \cap \mathcal{C}_{\mathbb{R}}(K)$

$f \in A \Longrightarrow f = u + iv$

$u, v \in \mathcal{C}_{\mathbb{R}}(K)$

$$f(x) = u(x) + iv(x)$$

$$\bar{f}(x) = u(x) - iv(x)$$

$$\Rightarrow \frac{1}{2}(f + \bar{f}) = u \quad \frac{1}{2i}(f - \bar{f}) = v$$

If $\mathcal{A}$ is self adjoint, $u, v \in \mathcal{A}_{\mathbb{R}}$.

$x_1, x_2 \in K \quad x_1 \neq x_2$

$$\exists f \in \mathcal{A} \quad f(x_1) \neq f(x_2)$$

$$u(x_1) + iv(x_2) \neq u(x_2) + iv(x_2)$$

Either $u(x_1) \neq u(x_2)$ or $v(x_1) \neq v(x_2)$

(a) holds for $\mathcal{A}_{\mathbb{R}}$.

$$f(x) \neq 0 \quad u(x) + iv(x) \neq 0$$

either $u(x) \neq 0$ or $v(x) \neq 0$

$\Rightarrow$ (b) holds

$\mathcal{A}_{\mathbb{R}}$ is dense in $\mathcal{C}_{\mathbb{R}}(K)$ $\Rightarrow$

$A$ is dense in $\mathcal{C}(K)$. ▨

Example

$\mathbb{R}$ - real line, $f$ continuous function on $\mathbb{R}$ with period $2\pi$ $\quad f(x) = f(x+2\pi)$

$$\Rightarrow \max_{x\in\mathbb{R}} |f(x)| = \max_{x\in[0,2\pi]} |f(x)|$$

$$\mathcal{C}_{per}(\mathbb{R}) \xrightarrow{res} \mathcal{C}_0([0,2\pi]).$$

$f(0)=f(2\pi)$

$$\downarrow$$

$$\mathcal{C}(S^1).$$

$e^{inx}$

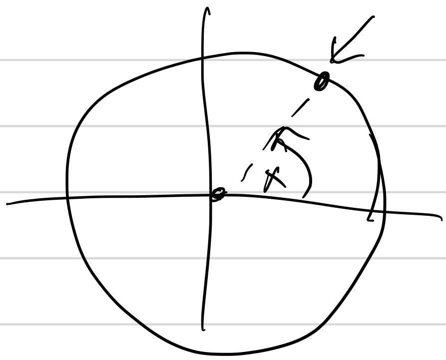
$$A = \left\{ \sum_{n=-N}^{N} a_n e^{inx} \;\middle|\; a_n \in \mathbb{C} \right\}$$

subalgebra of $\mathcal{C}_{per}(\mathbb{R})$.

A contains constants

① $f(x) \neq 0$ for $x \in S^1$.

② $x_1 \neq x_2$ $\quad e^{ix_1} \neq e^{ix_2}$



③ $\overline{e^{inx}} = e^{-inx}$.

A is dense in $\mathcal{C}_{per}(\mathbb{R})$.

$f$ a continuous periodic function on $\mathbb{R}$, $\varepsilon > 0$. There exists a trigonometric polynomial

$$P = \sum_{n=-N}^{N} a_n e^{inx}$$

such that

$$|f(x) - P(x)| < \varepsilon.$$

We want to analyse the trigonometric polynomials

$$\sum_{n=-N}^{N} a_n e^{inx}$$