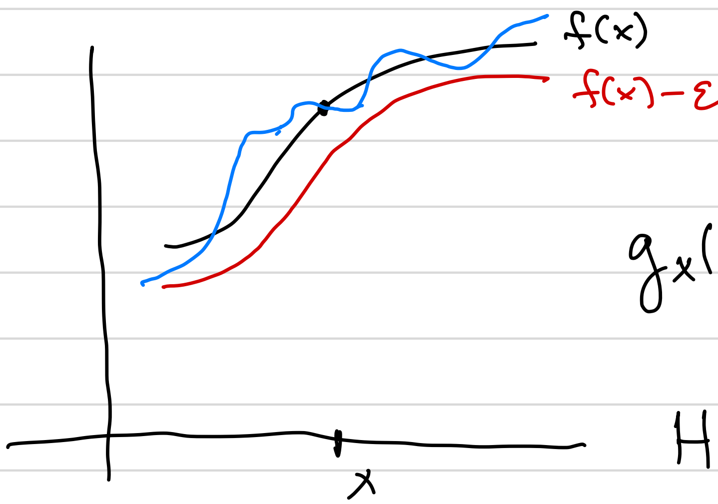
Last time we proved:

Let $f \in \mathcal{C}(K)$. For $x \in K$ there exists $g_x \in \bar{A}$ such that $f(x) = g_x(x)$ and $g_x(y) > f(y) - \varepsilon$ for all $y \in K$.



$$\underbrace{g_x(x) - f(x)}_{=0} < \varepsilon$$

Hence there exist an open neighborhood V_x of x such that

$$g_x(y) - f(y) < \varepsilon$$

for $y \in V_x$.

$(V_x; x \in K)$ is an open cover of K. There exists a finite subcover $(V_{x_1}, \dots, V_{x_\ell})$ of K

$$g = \min(g_{x_1}, \dots, g_{x_\ell}) \in \bar{A}$$

$$\Rightarrow \quad g(z) - f(z) < \varepsilon,$$

and

$$g(z) - f(x) > -\varepsilon.$$

This implies

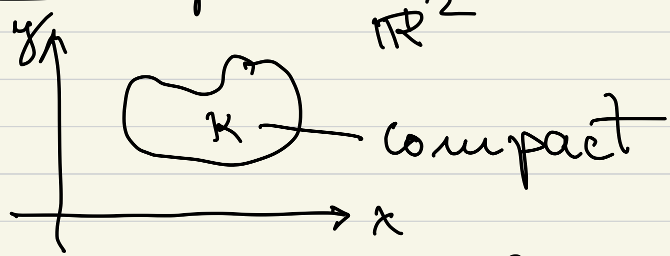
$$|f(x) - g(x)| < \varepsilon.$$

Since $g \in \bar{A}$, for any $\varepsilon > 0$ there exists $h \in A$ such that $\|g - h\| < \varepsilon$, i.e., $|g(x) - h(x)| < \varepsilon$.

$$\Rightarrow \quad |f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| < 2\varepsilon.$$

This completes the proof of Stone-Weierstrass.

Examples



$\mathcal{C}(K)$ algebra of continuous real valued functions

$$\|f\| = \max_{x \in K} |f(x)|$$

A - subalgebra of restrictions of polynomials in x and y

$$A = \left\{ \sum_{n,m=0}^{N} a_{n,m} x^n y^m \; ; \; N \in \mathbb{Z}_+ , \; a_{n,m} \in \mathbb{R} \right\}$$

closed under addition and multiplication.

A contains constants

- doesn't vanish at any $x \in K$.

$(x_1, y_1), (x_2, y_2) \in K$

$(x_1, y_1) \neq (x_2, y_2)$

$\Rightarrow$ either $x_1 \neq x_2$ or $y_1 \neq y_2$.

Hence polynomials

$(x,y) \longmapsto x$ or $(x,y) \longmapsto y$

differ points in K.

By Stone-Weierstrass theorem, for any $f \in \mathcal{C}(K)$, $\varepsilon > 0$, there exists a polynomial $P(x,y)$ such that

$$|f(x,y) - P(x,y)| < \varepsilon$$

for all $(x,y) \in K$.

<u>Complex version</u> : K compact space

$\mathcal{C}(K)$ – complex continuous functions on K

$$d(f,g) = \max_{x \in K} |f(x) - g(x)|$$

$A \subset \mathcal{C}(K)$ complex subalgebra

(a) $x_1, x_2 \in K \ \exists f \in A \ f(x_1) \neq f(x_2)$;

(b) $x \in K \ \exists f \in A \ f(x) \neq 0$

(c) $f \in A \Rightarrow \bar{f} \in A$ <u>selfadjoint</u>

<u>Preparation</u> $z \in \mathbb{C}$ $z = x + iy$

$x, y \in \mathbb{R}$ $|z| = \sqrt{x^2 + y^2} = \|(x,y)\|$

$d(z_1, z_2) = |z_1 - z_2|$ is a metric on $\mathbb{C}$. $\mathbb{C}$ is a topological space

$z = x + iy$ $\bar{z} = x - iy$ complex conjugate of z

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

$$|z| = (z\bar{z})^{1/2}.$$

$$f : K \longrightarrow \mathbb{C}$$

$$f(x) = u(x) + i v(x)$$

$u : K \longrightarrow \mathbb{R}$ $v : K \longrightarrow \mathbb{R}$

f continuous $\Longleftrightarrow$ u, v continuous