Assume that $\mathcal{A}$ satisfies (a) and (b).

$\overline{\mathcal{A}}$ closure of $\mathcal{A}$

$f \in \mathcal{A} \Longrightarrow |f| \in \overline{\mathcal{A}}$

$f \in \mathcal{A}$ P polynomial

$P(f) = a_0 + a_1 f + \dots + a_n f^n$

P has no constant term $\Longrightarrow$

$P(f) \in \mathcal{A}$.

f is bounded on K

$\Longrightarrow -M \leq f(x) \leq M$

By Weierstrass, there exists

$P_n \to |x|$ on $[-M, M]$

$\Rightarrow \exists\, n_0,\ n \geq n_0 \Longrightarrow |P_n(x) - |x|| < \frac{\varepsilon}{2}$

no constant term

$$\left| \underbrace{P_n(x) - P_n(0)}_{= Q_n(x)} - |x| \right| \le$$

$$\le \left| P_n(x) - |x| \right| + \left| P_n(0) \right| < \varepsilon$$

for all $x \in [-M, M]$.

$$\Longrightarrow \left| Q_n(f(x)) - |f(x)| \right| < \varepsilon$$

for all $x \in K$.

$Q_n(f) \in A$

$\Longrightarrow |f| \in \overline{A}$. $\blacksquare$

f, g in $\mathcal{C}(K)$

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$$

$$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$

$f, g \in A \Longrightarrow \max(f,g), \min(f,g) \in \overline{A}$.

Let $x_1 \neq x_2 \in K$ $c_1, c_2 \in \mathbb{R}$

<u>Claim</u>. There exists $f \in \mathcal{A}$ such that $f(x_1) = c_1$, $f(x_2) = c_2$.

<u>Proof</u>:

$\exists\, g \in \mathcal{A}$ $g(x_1) \neq g(x_2)$ $h(x_1) \neq 0$, $k(x_2) \neq 0$

$$u = gk - g(x_1)k$$

$$u(x_1) = 0 \qquad u(x_2) = g(x_2)k(x_2) - g(x_1)k(x_2) = (g(x_2) - g(x_1))k(x_2) \neq 0$$

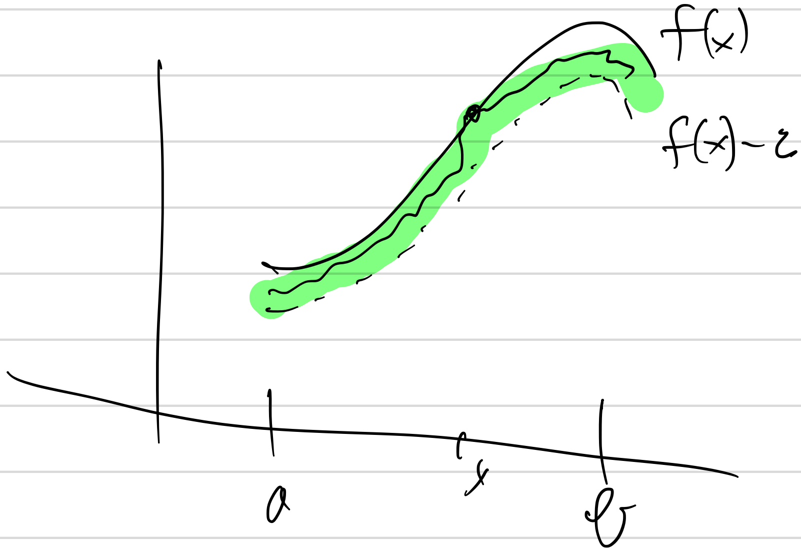
$$v = gh - g(x_2)h$$

$$v(x_2) = 0 \qquad v(x_1) = (g(x_1) - g(x_2))h(x_1) \neq 0$$

$$f = c_1 \frac{v}{v(x_1)} + c_2 \frac{u}{u(x_2)}$$

$$f(x_1) = c_1 \frac{v(x_1)}{v(x_1)} + c_2 \frac{0}{u(x_2)} = c_1$$

$$f(x_2) = c_1 \frac{\overbrace{v(x_2)}^{=0}}{v(x_1)} + c_2 \frac{u(x_2)}{u(x_2)} = c_2 .$$



<u>Lemma</u>. Let $f \in \mathcal{C}(K)$. Let $x \in K$ then there exists function $g_x \in \mathcal{A}$ such that $g_x(x) = f(x)$ and $g_x(y) > f(y) - \varepsilon$,

<u>Proof</u>. For any $y \in X$, $y \neq x$ there exists a function $h_y \in \mathcal{A}$ such that

$$h_y(x) = f(x) \qquad h_y(y) = f(y).$$

Since $h_y - f$ is continuous, $h_y(z) - f(z) > -\varepsilon$ in a neighborhood V_y of y.

Take a finite subcover $V_{y_1}, \dots, V_{y_m}$ of K

$$g_x = \max(h_{y_1}, \dots, h_{y_m})$$

$$g_x(x) = f(x) \qquad g_x(z) - f(z) > -\varepsilon$$

for all $z \in K$. Hence

$g_x(z) > f(z) - \varepsilon$. ∎