K - compact space

$\mathcal{C}(K)$ - real valued continuous functions on K

- normed algebra

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\|f\| = \max_{x \in X} |f(x)|$.

$\mathcal{A}$ subalgebra of $\mathcal{C}(K)$.

$\bar{\mathcal{A}}$ - closure of $\mathcal{A}$

$\bar{\mathcal{A}}$ is subalgebra of $\mathcal{C}(K)$

$\mathcal{A} \subset \bar{\mathcal{A}} \subset \mathcal{C}(K)$

To show this we have to show that $f, g \in \bar{\mathcal{A}} \Longrightarrow$

$f+g \in \overline{A}$, $f \cdot g \in \overline{A}$.

<u>Remark</u>. If $f \notin \overline{A}$, f is in $\mathcal{C}(K) \setminus \overline{A}$. This set is open and there exists $\varepsilon > 0$ such that

$$\{P \in \mathcal{C}(K) \mid \|f - P\| < \varepsilon\} \subset \mathcal{C}(K) \setminus \overline{A}.$$

Therefore, if $f \in \overline{A}$, for any $\varepsilon > 0$ there exists $P \in A$ $\|f - P\| < \varepsilon$.

Assume that $f, g \in \overline{A}$. Then there exist $P, Q \in A$

such that $\|f-P\|<\varepsilon/2$, $\|g-Q\|<\varepsilon/2$

$$\Rightarrow \|(f+g)-(P+Q)\| =$$

$$= \|(f-P)+(g-Q)\| \leq$$

$$\leq \|f-P\|+\|g-Q\| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

$\Rightarrow f+g \in \overline{A}$.

<u>Exercise</u>: Prove $f, g \in \overline{A}$

$\Rightarrow f \cdot g \in \overline{A}$.

Let X be a topological space, $Y \subset X$. If $\overline{Y} = X$, we say that Y is dense in X,

$\mathcal{A}$ is a subalgebra

$\tilde{\mathcal{A}}$ – is also a subalgebra

<u>Theorem</u> (Stone–Weierstrass)
Let $\mathcal{A}$ be a subalgebra in $\mathcal{C}(K)$. Assume that

(a) $\mathcal{A}$ separates points in K (i.e. $x_1, x_2 \in K \Longrightarrow \exists f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$)

(b) $\mathcal{A}$ vanishes at no point in K (i.e. for $x \in K$ there exists $f \in \mathcal{A}$ $f(x) \neq 0$).

Then $\mathcal{A}$ is dense in $\mathcal{C}(K)$

i.e. $\bar{\mathcal{A}} = \mathcal{C}(K)$.

<u>Explanation of Weierstrass thm.</u>

$\mathcal{P}$, $x_1, x_2 \in [a,b]$ $x_1 \neq x_2$

$P(x) = x$ $P(x_1) = x_1 \neq x_2 = P(x_2)$

$\mathcal{P}$ separates points

$1 \in \mathcal{P}$ $1(x) = 1$

$\mathcal{P}$ doesn't vanish at a pt. in $[a,b]$.

$\Rightarrow \mathcal{P}$ is dense in $\mathcal{C}([a,b])$

$\Rightarrow f \in \mathcal{C}([a,b])$

For any $\varepsilon > 0$ $\|f - P\| < \varepsilon$

polynomial

Assume that $\mathcal{A}$ satisfies (a) and (b).

$\overline{\mathcal{A}}$ closure of $\mathcal{A}$

$f \in \mathcal{A} \Longrightarrow |f| \in \overline{\mathcal{A}}$

$f \in \mathcal{A}$ P polynomial

$P(f) = a_0 1 + a_1 f + \dots + a_n f^n$

P has no constant term $\Longrightarrow$ $P(f) \in \mathcal{A}$.

f is bounded on K

$\Longrightarrow -M \leq f(x) \leq M$

By Weierstrass theorem for $\varepsilon > 0$ there exists a polynomial P such that $|P(x) - |x|| < \varepsilon/2$ for $x \in [-M, M]$.

$Q(x) = P(x) - P(0)$ – no constant term.

$$|Q(x) - |x|| = |P(x) - P(0) - |x|| \le$$
$$|P(x) - |x|| + |P(0) - |0|| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $x \in [-M, M]$.

Hence

$$|Q(f(x)) - |f(x)|| < \varepsilon \text{ for } x \in K.$$

$$\Rightarrow \|Q(f) - |f|\| < \varepsilon$$

$$Q(f) = a_1 f + a_2 f^2 + \dots + a_n f^n$$

$a_1, a_2, \dots, a_n \in \mathbb{R}$.

$$Q(f) \in A \Longrightarrow |f| \in \overline{A}.$$

Since, $\overline{\overline{A}} = \overline{A}$, it follows that $f \in \overline{A} \Rightarrow |f| \in \overline{A}$!