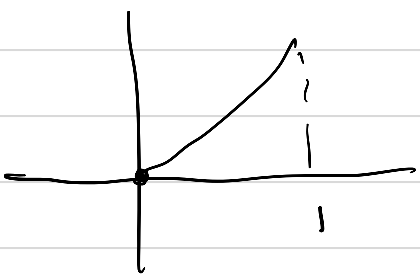
Consider

$$F(x) = (1-x^2)^n - (1-nx^2) \quad , \quad F(0) = 0$$

$$F'(x) = -n(1-x^2)^{n-1}\cdot 2x + 2nx =$$

$$= 2nx\left(1 - (1-x^2)^{n-1}\right) \geq 0$$

for $0 \leq x \leq 1$. F is increasing on $[0,1]$.



$F(x) \geq 0$, $x \in [0,1]$.

$$(1-x^2)^n \geq (1-nx^2) \quad \text{for } x \in [0,1].$$

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx \geq$$

$$2\int_0^{\frac{1}{\sqrt{n}}}(1-x^2)^n dx \geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-nx^2)\,dx$$

$$= 2\left[x - n\cdot\frac{x^3}{3}\right]_0^{\frac{1}{\sqrt{n}}} = 2\left[\frac{1}{\sqrt{n}} - \frac{n}{3}\,\frac{1}{n^{3/2}}\right] =$$

$$= \frac{2}{\sqrt{n}} - \frac{2}{3\sqrt{n}} = \frac{4}{3}\,\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}.$$

$$1 = \int_{-1}^{1} Q_n(x)\,dx =$$

$$= c_n \int_{-1}^{1} (1-x^2)^n\,dx \geq \frac{c_n}{\sqrt{n}}$$

$$\boxed{c_n \leq \sqrt{n}}$$

$$|Q_n(x)| \leq \sqrt{n}\,(1-x^2)^n$$

If $|x| \geq \delta$

$$\Rightarrow |Q_n(x)| \leq \sqrt{n}\,(1-\delta^2)^n$$

$Q_n(x) \to 0$ uniformly on $[\delta, 1]$.

This completes the proof of ($*$) and the proof of Weierstrass theorem.

Stone - Weierstrass thm.

K compact space

$\mathcal{C}(K)$ is an algebra of functions

$+$ addition of functions

$\cdot$ mult of functions

$+$ is associative and commutative
 - ablian group

$\cdot$ is associative and commutative

distributivity

- commutative ring
- vector space over $\mathbb{R}$
- commutative algebra over $\mathbb{R}$
- has identity 1

(compare with matrices).

$\mathcal{A}$ is a subalgebra if $f, g \in \mathcal{A} \Rightarrow f+g, f \cdot g, \alpha f, \alpha \in \mathbb{R}$ are in $\mathcal{A}$.

Example. $K = [a,b]$
$\mathcal{C}(K)$ is an algebra over $\mathbb{R}$
$\mathcal{P}$ - polynomials on $[a,b]$ are a subalgebra of $\mathcal{C}([a,b])$

Generalization of Weierstrass thm.

Question: When is $\mathcal{A}$ dense in $\mathcal{C}(K)$?