Topology

Let S and T be two sets and $f : S \longrightarrow T$ a function. To study f we have to specify its properties like "continuity" and "differentiability". To do this we need to define additional structure on the sets S and T to be able to formulate notions like "points near to each other"...

The first attempt is to define metric spaces

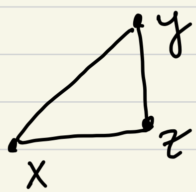
X - set $\quad d : X \times X \longrightarrow \mathbb{R}$

function on $X \times X$ with values in $\mathbb{R}$

1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \iff x = y$
3. $d(x,y) = d(y,x)$ (symmetry)
4. $x, y, z \in X$

$$d(x,y) \leq d(x,z) + d(z,y)$$

– triangle inequality



Such function $d : X \times X \to \mathbb{R}$ is called a metric or distance on X and the pair (X, d) is called a metric space.

Examples:

① $X = \mathbb{R}$, $d(x,y) = |x-y|$

② $X = \mathbb{R}^2$, $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

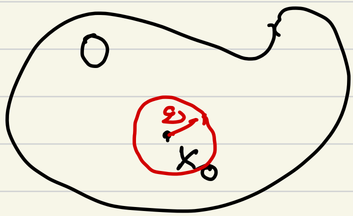
Exercise: Check that d is a metric.

Two points x, y in a metric space X are "near" to each other if their distance $d(x,y)$ is small.

A ball centered at x_0 of radius $\varepsilon > 0$ is

$$B(x_0, \varepsilon) = \{ y \in X \mid d(y, x_0) < \varepsilon \}$$

A subset $O \subset X$ is a neighborhood of x_0 if it contains a ball centered at x_0.



A set $U \subset X$ is <u>open</u> if it is a neighborhood of all of its points.

<u>Example</u>: $B(x_0, \varepsilon)$ is an open set.

Let $y \in B(x_0, \varepsilon)$, $d(x_0, y) = \delta < \varepsilon$.

Let $z \in B(y, \varepsilon - \delta)$.

Then $d(y, z) < \varepsilon - \delta$.

Hence, we have

$$d(x_0, z) \leq d(x_0, y) + d(y, z) < \delta + \varepsilon - \delta = \varepsilon,$$

and $z \in B(x_0, \varepsilon)$.

It follows that $B(y, \varepsilon-\delta) \subset B(x_0, \varepsilon)$.
Hence, $B(x_0, \varepsilon)$ is an open set in X.
So we shall call it an <u>open</u> ball.

<u>Warning</u>. Empty set is open. It contains no points so the above condition is vacuous.

X is open – trivial.

<u>Fact</u>. Union of a family of open sets is open.

<u>Proof</u>. Let U_i, $i \in I$, be a family of open sets.

index set

Put $U = \bigcup_{i \in I} U_i$.

$x_0 \in U \implies x_0 \in U_i$ for some $i \in I$. $\exists \varepsilon > 0$ such that $B(x_0, \varepsilon) \subset U_i \implies B(x_0, \varepsilon) \subset U$.

$\blacksquare$

<u>Fact</u>. Intersection of <u>finitely many</u> open sets is open.

<u>Proof</u>. If the intersection is empty we are done.

If $x_0 \in U_1 \cap U_2 \cap \ldots \cap U_n$, $x_0 \in U_i$, $1 \leq i \leq n$, $\exists\ \varepsilon_i > 0$ such that $B(x_0, \varepsilon_i) \subset U_i$

Put $\varepsilon = \min_{1 \leq i \leq \varepsilon} \varepsilon_i$. Then

$$B(x_0,\varepsilon) \subset B(x_0,\varepsilon_i) \subset U_i$$

$$\Rightarrow B(x_0,\varepsilon) \subset U_1 \cap U_2 \cap \dots \cap U_n .$$

<u>Warning</u>. This is false for intersections of infinite families.

<u>Example</u>: $X = \mathbb{R}$ $d(x,y) = |x-y|$

$U_n = (-\frac{1}{n}, \frac{1}{n})$ – are open sets

actually balls centered at 0

of radius $\frac{1}{n}$.

$$\bigcap_{n=0}^{\infty} U_n = \{0\}$$

$\{0\}$ is <u>not</u> open since it does not contain a ball.

<u>Example</u>. X set

$d : X \times X \to \mathbb{R}$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

This is called the <u>discrete metric</u>

(check that it is a metric).

$B(x_0, \varepsilon) = X$ if $\varepsilon > 1$.

$B(x_0, \varepsilon) = \{x_0\}$ if $0 < \varepsilon \leq 1$.

Any subset of X is open!