Math 3210-02 Midterm 2, November 26, 2019

Solutions

Problem 1 (25 points)**.** Let

$$f(x) = \frac{\sin x}{|x|}$$

for $x \neq 0$. Does $\lim_{x\to 0} f(x)$ exist? Explain your answer!

Solution: By L'Hôspital rule we have

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1.$$

Hence the limit from the right of $\frac{\sin x}{|x|}$ at 0 is equal to 1.

On the other hand the limit from the left at 0 (since x is negative there) is equal to the limit of

$$\frac{\sin x}{|x|} = -\frac{\sin x}{x}$$

at 0, i.e., it is equal to -1.

Therefore the limit at 0 does not exist!

Problem 2 (25 points)**.** Show that the function $\ln x$ is uniformly continuous on $[1, +\infty)$.

Solution: The derivative of $\ln x$ is equal to $\frac{1}{x}$. So, it is ≤ 1 on $[1, +\infty)$. Hence, the statement follows by applying Theorem 4.3.9.

Problem 3 (25 points)**.** Suppose f is any non-decreasing function on a bounded interval $[a, b]$. If P_n is a partition of $[a, b]$ into n equal subintervals, show that

$$U(f, P_n) - L(f, P_n) = (f(b) - f(a))\frac{b-a}{n}.$$

What do you conclude about the integrability of f?

Solution: Since the function is non-decreasing, for any interval $[x_{i-1}, x_i]$ in a partition P we have

$$f(x_{i-1}) \leq f(x) \leq f(x_i)$$

for all $x \in [x_{i-1}, x_i]$. Therefore, we have $f(x_{i-1}) = \inf_{x\in[x_{i-1},x_i]} f(x)$ and $f(x_i) = \sup_{x\in[x_{i-1},x_i]} f(x)$. Hence, we have

$$U(f,P) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(f,P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

for any partition P. It follows that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}).$$

Since for our partitions P_n we have $x_i - x_{i-1} = \frac{b-a}{n}$, it follows that

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{b-a}{n} \left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(x_{i-1}) \right) = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Therefore, for large n, the difference $U(f,P_n) - L(f,P_n)$ can be made arbitrarily small, i.e., f is integrable.

Problem 4 (25 points)**.** Find

$$\frac{d}{dx} \int_{\frac{1}{x}}^{x} e^{-t^2} dt.$$

Solution: The function e^{-x^2} is continuous. Therefore, by Second Fundamental Theorem of Calculus, the function

$$F(x) = \int_0^x e^{-t^2}\, dt$$

is differentiable and its derivative is

$$F'(x) = e^{-x^2}.$$

Our function is

$$G(x) = \int_{\frac{1}{x}}^{x} e^{-t^2}\,dt = \int_0^x e^{-t^2}\,dt - \int_0^{\frac{1}{x}} e^{-t^2}\,dt = F(x) - F\left(\frac{1}{x}\right).$$

Hence, by Chain Rule we have

$$G'(x) = F'(x) - F'\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) = e^{-x^2} + \frac{e^{-\frac{1}{x^2}}}{x^2}.$$