Math 3210-02 Midterm 2, November 26, 2019

Solutions

Problem 1 (25 points). Let

\[ f(x) = \frac{\sin x}{|x|} \]

for \( x \neq 0 \). Does \( \lim_{x \to 0} f(x) \) exist? Explain your answer!

Solution: By L’Hôpital rule we have

\[ \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \cos x = 1. \]

Hence the limit from the right of \( \frac{\sin x}{|x|} \) at 0 is equal to 1.

On the other hand the limit from the left at 0 (since \( x \) is negative there) is equal to the limit of

\[ \frac{\sin x}{|x|} = -\frac{\sin x}{x} \]

at 0, i.e., it is equal to \(-1\).

Therefore the limit at 0 does not exist!

Problem 2 (25 points). Show that the function \( \ln x \) is uniformly continuous on \([1, +\infty)\).

Solution: The derivative of \( \ln x \) is equal to \( \frac{1}{x} \). So, it is \( \leq 1 \) on \([1, +\infty)\). Hence, the statement follows by applying Theorem 4.3.9.

Problem 3 (25 points). Suppose \( f \) is any non-decreasing function on a bounded interval \([a, b]\). If \( P_n \) is a partition of \([a, b]\) into \( n \) equal subintervals, show that

\[ U(f, P_n) - L(f, P_n) = (f(b) - f(a)) \frac{b - a}{n}. \]

What do you conclude about the integrability of \( f \)?
Solution: Since the function is non-decreasing, for any interval \([x_{i-1}, x_i]\) in a partition \(P\) we have
\[
f(x_{i-1}) \leq f(x) \leq f(x_i)
\]
for all \(x \in [x_{i-1}, x_i]\). Therefore, we have \(f(x_{i-1}) = \inf_{x \in [x_{i-1}, x_i]} f(x)\) and \(f(x_i) = \sup_{x \in [x_{i-1}, x_i]} f(x)\). Hence, we have
\[
U(f, P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})
\]
and
\[
L(f, P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})
\]
for any partition \(P\). It follows that
\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}).
\]
Since for our partitions \(P_n\) we have \(x_i - x_{i-1} = \frac{b-a}{n}\), it follows that
\[
U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \frac{b-a}{n} (f(b) - f(a)).
\]
Therefore, for large \(n\), the difference \(U(f, P_n) - L(f, P_n)\) can be made arbitrarily small, i.e., \(f\) is integrable.

Problem 4 (25 points). Find
\[
\frac{d}{dx} \int_{\frac{x}{2}}^{x} e^{-t^2} dt.
\]

Solution: The function \(e^{-x^2}\) is continuous. Therefore, by Second Fundamental Theorem of Calculus, the function
\[
F(x) = \int_{0}^{x} e^{-t^2} dt
\]
is differentiable and its derivative is
\[
F'(x) = e^{-x^2}.
\]
Our function is
\[ G(x) = \int_{\frac{1}{x}}^{x} e^{-t^2} \, dt = \int_{0}^{x} e^{-t^2} \, dt - \int_{0}^{\frac{1}{x}} e^{-t^2} \, dt = F(x) - F\left(\frac{1}{x}\right). \]

Hence, by Chain Rule we have
\[ G'(x) = F'(x) - F'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = e^{-x^2} + \frac{e^{-\frac{1}{x^2}}}{x^2}.\]