Math 3210-4 Take Home Midterm 2, October 28, 2017

Show all work!

Name:

Problem 1. Let $X$ be a metric space and $K$ a compact subset of $X$. Let $x$ be a point outside $K$. Show that there exists a point $y \in K$ closest to $x$.

Problem 2. If $f$ is a continuous mapping of a topological space $X$ into a topological space $Y$, prove that $f(\bar{E}) \subset \overline{f(E)}$ for every subset $E$ of $X$. Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

Problem 3. Consider the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$ 

(i) Show that its radius of convergence is 1.

(ii) Is the power series convergent at $z = 1$?

(ii) Is the power series convergent at $z = -1$?

Problem 4. Show that the power series

(i)

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

(ii)

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converge for all complex $z$.

Define continuous functions

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
for all \( z \in \mathbb{C} \).

Show that
\[
\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}
\]
and
\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2}
\]
for all \( z \in \mathbb{C} \).

**Problem 5.** Prove the Euler formula
\[
e^{ix} = \cos(x) + i\sin(x)
\]
for all \( x \in \mathbb{R} \).

Use this to show that for any \( z = x + iy, x, y \in \mathbb{R} \), we have
\[
e^z = e^x(\cos(y) + i\sin(y)).
\]

**Problem 6.** Prove the following identities
(i)
\[
\sin(z)^2 + \cos(z)^2 = 1
\]
for all \( z \in \mathbb{C} \);
(ii)
\[
\sin(u + v) = \sin(u)\cos(v) + \cos(u)\sin(v)
\]
for all \( u, v \in \mathbb{C} \);
(ii)
\[
\cos(u + v) = \cos(u)\cos(v) - \sin(u)\sin(v)
\]
for all \( u, v \in \mathbb{C} \).