Problem 1 (25 points). Prove by induction that
\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \]

**Solution:** If \( n = 1 \), both sides are equal to 1. Assume that the identity holds for \( n \). Then, by induction assumption, we have
\[
\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2
= (n+1)\left(\frac{n(2n+1)+6(n+1)}{6}\right) = (n+1)\frac{2n^2 + 7n + 6}{6}
= \frac{(n+1)(n+2)(2n+3)}{6}.
\]
Hence the statement follows by induction.

Problem 2 (25 points). Find
\[ \lim_{n \to \infty} \frac{2n^3 - 5}{n^3 + 2n^2 + 5}. \]
Explain your reasoning.

**Solution:** We have
\[ \frac{2n^3 - 5}{n^3 + 2n^2 + 5} = \frac{2 - \frac{5}{n^3}}{1 + \frac{2}{n} + \frac{5}{n^3}}. \]
Since \( \frac{1}{n} \to 0 \) we see that denominator tends to 1 and numerator tends to 2. Hence the limit is 2.

Problem 3 (25 points). Let \( I = [0, 1] \) and \( f \) and \( g \) two continuous functions on \( I \). Assume that \( f(0) < g(0) \) and \( f(1) > g(1) \). Show that there exists a point \( c \in [0, 1] \) such that \( f(c) = g(c) \).
Solution: Consider the function $h(x) = f(x) - g(x)$ on $I$. This function is continuous. Moreover, $h(0) = f(0) - g(0) < 0$ and $h(1) = f(1) - g(1) > 0$. Therefore, by the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that $h(c) = 0$. This implies that $f(c) = g(c)$.

Problem 4 (25 points). Let

$$f_n(x) = \sin\left(\frac{x}{n}\right)$$

for all $x \in \mathbb{R}$. Show that the sequence of functions $(f_n)$ converges pointwise to 0, but does not converge uniformly on $\mathbb{R}$.

Solution: For any $x$ we have $\frac{x}{n} \to 0$ as $n \to \infty$. Since $\sin$ is continuous on $\mathbb{R}$, this implies that

$$\lim f_n(x) = \lim \sin\left(\frac{x}{n}\right) = \sin(0) = 0$$

for any $x$. On the other hand, for any $n$,

$$f_n\left(\frac{n\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

So, $f_n$ cannot converge uniformly to 0!