Math 3210-02: Homework 6, November 25, 2019

Solutions

Problem 1 (20 points)**.** Determine if the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k + k - 1}$$

converges or diverges.

Solution: Clearly, we have

$$2^k + k - 1 \geq 2^k$$

for all k. Therefore, we have

$$\frac{1}{2^k + k - 1} \leq \frac{1}{2^k}$$

for all k. Since

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

converges, our series also converges by Comparison Test.

Problem 2 (20 points)**.** Prove that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{2^k}$$

is defined and continuous on the entire real line.

Solution: Clearly, for any real x we have

$$\left| \frac{\sin(kx)}{2^k} \right| \leq \frac{1}{2^k}$$

for all k. By Weierstrass M-test, the series converges uniformly for all x. Therefore, the sum of the series is a continuous function by Theorem 6.4.2.

Problem 3 (20 points)**.** Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} 2^k x^{2k}.$$

Solution: We have

$$2^k x^{2k} = (2x^2)^k$$

for all k, so our series is a geometric series converging for $2x^2 < 1$ and diverging for $2x^2 > 1$. It follows that our series converges for $|x| < \frac{1}{\sqrt{2}}$ and diverges for $|x| > \frac{1}{\sqrt{2}}$. Hence the radius of convergence is $\frac{1}{\sqrt{2}}$.

Problem 4 (20 points)**.** Find a power series on $(-1, 1)$ which converges to

$$\frac{1}{(1-x)^3}.$$

Solution: We have the sum of geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which converges on $(-1, 1)$. By differentiation we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

on $(-1, 1)$. Differentiating again, we get

$$\frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} (n+1) n x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) x^n$$

on $(-1, 1)$. Hence we have

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$$

on $(-1, 1)$.

Problem 5 (20 points)**.** The function *hyperbolic cosine* is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

find its Taylor series for $a = 0$. Show that the series converges to the function on the entire real line.

Solution: Taylor series for e^x are given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and they converge for all real x. Therefore, we have

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

for all real x. It follows that

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

for all real x.