Problem 1 (20 points). Prove that
\[ 1 \leq \int_{-1}^{1} \frac{1}{1 + x^{2n}} \, dx \leq 2 \]
for all \( n \in \mathbb{N} \).

**Solution:** Since \( 0 \leq x^{2n} \leq 1 \) for \( x \in [-1, 1] \), we have
\[ 1 \leq 1 + x^{2n} \leq 2 \]
and
\[ \frac{1}{2} \leq \frac{1}{1 + x^{2n}} \leq 1 \]
for \( x \in [-1, 1] \). This implies that
\[ 1 = \int_{-1}^{1} \frac{1}{2} \, dx \leq \int_{-1}^{1} \frac{1}{1 + x^{2n}} \, dx \leq \int_{-1}^{1} \, dx = 2. \]

Problem 2 (20 points). Let \( \{f_n\} \) be a sequence of integrable functions defined on a closed bounded interval \([a, b]\). If \( \{f_n\} \) converges uniformly on \([a, b]\) to a function \( f \), prove that \( f \) is integrable and
\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx. \]

**Solution:** Let \( \epsilon > 0 \). Since \( \{f_n\} \) converge uniformly to \( f \), there exists \( N \) such that \( |f(x) - f_n(x)| < \epsilon \) for all \( x \in [a, b] \) for \( n \geq N \). Therefore,
\[ f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon \]
for all \( x \in [a, b] \). This implies that we have
\[ \inf_{x \in [x_{i-1}, x_i]} f_n(x) - \epsilon \leq \frac{1}{2} \, dx \leq \sup_{x \in [x_{i-1}, x_i]} f_n(x) + \epsilon \]
for all \( x \in [x_{i-1}, x_i] \). Moreover, we have
\[ \inf_{x \in [x_{i-1}, x_i]} f_n(x) - \epsilon \leq \inf_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f_n(x) + \epsilon \]
for all \( x \in [x_{i-1}, x_i] \). If \( P \) is a partition of \([a, b]\), this implies that
\[
\inf_{x \in [x_{i-1}, x_i]} f_n(x)(x_{i+1} - x_i) - \epsilon(x_{i+1} - x_i) \leq \inf_{x \in [x_{i-1}, x_i]} f(x)(x_{i+1} - x_i)
\leq \sup_{x \in [x_{i-1}, x_i]} f(x)(x_{i+1} - x_i) \leq \sup_{x \in [x_{i-1}, x_i]} f_n(x)(x_{i+1} - x_i) + \epsilon(x_{i+1} - x_i).
\]
Summing over \( i \) we get
\[
L(f_n, P) - \epsilon(b - a) \leq L(f, P) \leq U(f, P) \leq U(f_n, P) + \epsilon(b - a).
\]
It follows that
\[
U(f, P) - L(f, P) \leq U(f_n, P) - L(f_n, P) + 2\epsilon(b - a)
\]
for any \( n \geq N \). Since \( f_n \) are integrable, we can find \( P \) such that
\[ U(f_n, P) - L(f_n, P) < \epsilon. \]
Therefore, we get
\[ U(f, P) - L(f, P) < \epsilon(2(b - a) + 1). \]
Since \( \epsilon \) is arbitrary, we see that \( f \) is integrable. Now we have
\[
\left| \int_a^b f(x) \, dx - \int_a^b f_n(x) \, dx \right| \leq \left| \int_a^b (f(x) - f_n(x)) \, dx \right|
\leq \int_a^b |f(x) - f_n(x)| \, dx \leq \epsilon(b - a)
\]
for \( n \geq N \). This implies that
\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.
\]

**Problem 3** (20 points). Find
\[
\frac{d}{dx} \int_0^{2x} \sin t^2 \, dt.
\]

**Solution:** Let
\[
F(x) = \int_0^x \sin t^2 \, dt.
\]
Since \( \sin(t^2) \) is continuous function, by Second Fundamental Theorem of Calculus, \( F \) is a differentiable function and \( F'(x) = \sin(x^2) \). Hence, by chain rule we have
\[
\frac{d}{dx} F(2x) = F'(2x) \cdot 2 = 2 \sin(4x^2).\]
Problem 4 (20 points). Let \( f \) be a continuous function on the interval \([0, 1]\). Express
\[
\int_0^{\frac{\pi}{2}} f(\sin \theta) \cos \theta \, d\theta
\]
as an integral involving only the function \( f \).

Solution: The function \( \sin \theta \) is differentiable and its derivative \( \cos \theta \) is continuous, and therefore integrable on \([0, \frac{\pi}{2}]\). The image of \([0, \frac{\pi}{2}]\) under \( \sin \) is \([0, 1]\). Therefore, by the change of variables formula, we have
\[
\int_0^{\frac{\pi}{2}} f(\sin \theta) \cos \theta \, d\theta = \int_0^1 f(u) \, du.
\]

Problem 5 (20 points). Prove that
\[
\ln \left( \frac{a}{b} \right) = \ln a - \ln b
\]
for all \( a, b \in (0, +\infty) \).

Solution: Let \( x, y > 0 \). Then we have \( \ln(xy) = \ln x + \ln y \). Therefore, for \( x > 0 \), we have
\[
0 = \ln(1) = \ln \left( x \cdot \frac{1}{x} \right) = \ln(x) + \ln \left( \frac{1}{x} \right),
\]
and
\[
\ln \left( \frac{1}{x} \right) = -\ln x.
\]
This implies that
\[
\ln \left( \frac{a}{b} \right) = \ln \left( a \cdot \frac{1}{b} \right) = \ln a + \ln \left( \frac{1}{b} \right) = \ln a - \ln b
\]
for \( a, b > 0 \).