Math 3210-02: Homework 5, November 8, 2019

Solutions

Problem 1 (20 points)**.** Prove that

$$1 \leq \int_{-1}^{1} \frac{1}{1+x^{2n}}\, dx \leq 2$$

for all $n \in \mathbb{N}$.

Solution: Since $0 \leq x^{2n} \leq 1$ for $x \in [-1,1]$, we have

$$1 \leq 1 + x^{2n} \leq 2$$

and

$$\frac{1}{2} \leq \frac{1}{1+x^{2n}} \leq 1$$

for $x \in [-1,1]$. This implies that

$$1 = \int_{-1}^{1} \frac{1}{2}\, dx \leq \int_{-1}^{1} \frac{1}{1+x^{2n}}\, dx \leq \int_{-1}^{1} dx = 2.$$

Problem 2 (20 points)**.** Let $\{f_n\}$ be a sequence of integrable functions defined on a closed bounded interval $[a,b]$. If $\{f_n\}$ converges uniformly on $[a,b]$ to a function f, prove that f is integrable and

$$\int_a^b f(x)\, dx = \lim \int_a^b f_n(x)\, dx.$$

Solution: Let $\epsilon > 0$. Since $\{f_n\}$ converge uniformly to f, there exists N such that $|f(x) - f_n(x)| < \epsilon$ for all $x \in [a,b]$ for $n \geq N$. Therefore,

$$f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon$$

for all $x \in [a,b]$. This implies that we have

$$\inf_{x \in [x_{i-1}, x_i]} f_n(x) - \epsilon \leq f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f_n(x) + \epsilon$$

for all $x \in [x_{i-1}, x_i]$. Moreover, we have

$$\inf_{x \in [x_{i-1}, x_i]} f_n(x) - \epsilon \leq \inf_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f_n(x) + \epsilon$$

for all $x \in [x_{i-1}, x_i]$. If P is a partition of $[a, b]$, this implies that

$$\inf_{x\in[x_{i-1},x_i]} f_n(x)(x_{i+1} - x_i) - \epsilon(x_{i+1} - x_i) \leq \inf_{x\in[x_{i-1},x_i]} f(x)(x_{i+1} - x_i)$$
$$\leq \sup_{x\in[x_{i-1},x_i]} f(x)(x_{i+1} - x_i) \leq \sup_{x\in[x_{i-1},x_i]} f_n(x)(x_{i+1} - x_i) + \epsilon(x_{i+1} - x_i).$$

Summing over i we get

$$L(f_n, P) - \epsilon(b - a) \leq L(f, P) \leq U(f, P) \leq U(f_n, P) + \epsilon(b - a).$$

It follows that

$$U(f, P) - L(f, P) \leq U(f_n, P) - L(f_n, P) + 2\epsilon(b - a)$$

for any $n \geq N$. Since f_n are integrable, we can find P such that $U(f_n, P) - L(f_n, P) < \epsilon$. Therefore, we get

$$U(f, P) - L(f, P) < \epsilon(2(b - a) + 1).$$

Since ϵ is arbitrary, we see that f is integrable. Now we have

$$\left| \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \right| \leq \left| \int_a^b (f(x) - f_n(x))\, dx \right|$$
$$\leq \int_a^b |f(x) - f_n(x)|\, dx \leq \epsilon(b - a)$$

for $n \geq N$. This implies that

$$\lim \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Problem 3 (20 points)**.** Find

$$\frac{d}{dx} \int_0^{2x} \sin t^2\, dt.$$

Solution: Let

$$F(x) = \int_0^x \sin t^2\, dt.$$

Since $\sin(t^2)$ is continuous function, by Second Fundamental Theorem of Calculus, F is a differentiable function and $F'(x) = \sin(x^2)$. Hence, by chain rule we have

$$\frac{d}{dx} F(2x) = F'(2x) \cdot 2 = 2\sin(4x^2).$$

Problem 4 (20 points)**.** Let f be a continuous function on the interval $[0, 1]$. Express

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta\, d\theta$$

as an integral involving only the function f.

Solution: The function $\sin\theta$ is differentiable and its derivative $\cos\theta$ is continuous, and therefore integrable on $[0, \frac{\pi}{2}]$. The image of $[0, \frac{\pi}{2}]$ under sin is $[0, 1]$. Therefore, by the change of variables formula, we have

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta\, d\theta = \int_0^1 f(u)\, du.$$

Problem 5 (20 points)**.** Prove that

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

for all $a, b \in (0, +\infty)$.

Solution: Let $x, y > 0$. Then we have $\ln(xy) = \ln x + \ln y$. Therefore, for $x > 0$, we have

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right),$$

and

$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

This implies that

$$\ln\left(\frac{a}{b}\right) = \ln\left(a \cdot \frac{1}{b}\right) = \ln a + \ln\left(\frac{1}{b}\right) = \ln a - \ln b$$

for $a, b > 0$.