Problem 1 (20 points). Prove that

\[ 1 \leq \int_{-1}^{1} \frac{1}{1 + x^{2n}} \, dx \leq 2 \]

for all \( n \in \mathbb{N} \).

Problem 2 (20 points). Let \( \{f_n\} \) be a sequence of integrable functions defined on a closed bounded interval \([a, b]\). If \( \{f_n\} \) converges uniformly on \([a, b]\) to a function \( f \), prove that \( f \) is integrable and

\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx. \]

Problem 3 (20 points). Find

\[ \frac{d}{dx} \int_{0}^{2x} \sin t^2 \, dt. \]

Problem 4 (20 points). Let \( f \) be a continuous function on the interval \([0, 1]\). Express

\[ \int_{0}^{\pi} f(\sin \theta) \cos \theta \, d\theta \]

as an integral involving only the function \( f \).

Problem 5 (20 points). Prove that

\[ \ln \left( \frac{a}{b} \right) = \ln a - \ln b \]

for all \( a, b \in (0, +\infty) \).