Math 3210-02: Homework 4, October 25, 2019

Solutions

Problem 1 (20 points)**.** Using the definition of the derivative find that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$.

Solution: Let $y(x) = \frac{1}{x}$. Then

$$y'(x_0) = \lim_{x\to x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} = \lim_{x\to x_0} \frac{\frac{x_0 - x}{x x_0}}{x - x_0}$$
$$= -\lim_{x\to x_0} \frac{1}{x x_0} = -\frac{1}{x_0} \frac{1}{\lim_{x\to x_0} x} = -\frac{1}{x_0^2}.$$

Problem 2 (20 points)**.** Using the formula $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and formulas for derivatives for sin and cos find the formula for derivative of tan.

Using the formula for derivative of tan find the derivative of its inverse function $\tan^{-1}$.

Solution: First we have, for $f(x) = \tan x$, using quotient rule

$$f'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos^2(x)}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

Let $g(x) = \tan^{-1}(x)$ be the inverse function of f. By the formula for the derivative of the inverse function, we have

$$g'(x) = \frac{1}{\frac{1}{\cos^2(g(x))}} = \cos^2(g(x)).$$

On the other hand, we have

$$\cos^2(y) = \frac{\cos^2(y)}{\cos^2(y) + \sin^2(y)} = \frac{1}{1 + \tan^2(y)}.$$

It follows that

$$g'(x) = \frac{1}{1 + \tan^2(g(x))} = \frac{1}{1 + (\tan(\tan^{-1}(x)))^2} = \frac{1}{1 + x^2}.$$

Problem 3 (20 points)**.** Let f, g, and h be three differentiable functions. Find the formula for the derivative of $(f \circ g \circ h)(x) = f(g(h(x)))$. (Hint: Use the chain rule!)

Solution: By chain rule we have

$$\begin{aligned}(f \circ g \circ h)'(x) &= ((f \circ g) \circ h)'(x) \\ &= (f \circ g)'(h(x)) \cdot h'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).\end{aligned}$$

Problem 4 (20 points)**.** Prove that

$$|\sin(x) - \sin(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

Solution: By the Mean Value Theorem, we have

$$\frac{\sin(x) - \sin(y)}{x - y} = \sin'(c) = \cos(c)$$

for some $x \leq c \leq y$. Since $|\cos(c)| \leq 1$, we have

$$|\sin(x) - \sin(y)| \leq |x - y|.$$

Problem 5 (20 points)**.** Find

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3}.$$

Solution: By L'Hôspital's rule we have

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} = \frac{1}{3} \lim_{x \to 0} \frac{\cos(x) - 1}{x^2}.$$

Since the limits of numerator and denominator are 0, we have to apply the rule again to get

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = -\frac{1}{2} \lim_{x \to 0} \frac{\sin(x)}{x}.$$

Again we have that the limits of numerator and denominator are 0, and by applying the rule for the third time we get

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \cos(x) = 1.$$

Therefore, putting everything together we have

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}.$$