Math 3210-02: Homework 3, October 1, 2019

Solutions

Problem 1 (20 points). Is the function \( f(x) = \sin\left(\frac{1}{x}\right) \) continuous on \((0, 1)\)? Is it uniformly continuous on \((0, 1)\)? Justify your answers.

Solution: Since \( x \mapsto \sin(x) \) is continuous on \( \mathbb{R} \) and \( x \mapsto \frac{1}{x} \) is continuous on \( \mathbb{R} \setminus \{0\} \), the function \( f \) is continuous on \((0, 1)\).

Take sequences \((x_n)\) and \((y_n)\) given by \( x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \), and \( y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} \), in \((0, 1)\). Then we have

\[
0 \leq x_n - y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} - \frac{1}{\frac{3\pi}{2} + 2n\pi} = \frac{\pi}{(\frac{\pi}{2} + 2n\pi)(\frac{3\pi}{2} + 2n\pi)} \leq \frac{1}{4n^2\pi}
\]

and \( |x_n - y_n| \to 0 \) as \( n \to \infty \).

On the other hand, we have

\[
f(x_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1
\]

and

\[
f(y_n) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1.
\]

Hence

\[
|f(x_n) - f(y_n)| = 2,
\]

and function \( f \) cannot be uniformly continuous.

Problem 2 (20 points). Is the function \( f(x) = x \sin\left(\frac{1}{x}\right) \) uniformly continuous on \((0, 1)\)? Explain your answer!

Solution: As in the solution of Problem 1, we conclude that \( f \) is continuous on \((0, 1)\). Moreover, we can extend it to \([0, 1]\) by defining \( f(0) = 0 \). By the solution of Problem 3 in Homework 2 we know that this extended function is continuous on \([0, 1]\). Therefore, \( f \) is uniformly continuous by Theorem 3.3.4.

Problem 3 (20 points). Prove that the sequence \( \frac{\sin(nx)}{n} \) converges uniformly to 0 on \([0, 1]\).
Solution: We have
\[\left| \frac{\sin(nx)}{n} \right| \leq \frac{|\sin(nx)|}{n} \leq \frac{1}{n}\]
for all \(x\). Let \(\epsilon > 0\). Then there exists \(N > 0\) such that \(n > N\) implies that \(1/n < \epsilon\) and
\[\left| \frac{\sin(nx)}{n} \right| < \epsilon\]
for all \(x \in [0, 1]\).

Problem 4 (20 points). Let \(I = (a, b)\) be an open bounded interval. Let \(f\) be a uniformly continuous function on \(I\). Show that \(f\) is bounded.

Solution: Since \(f\) is uniformly continuous on \((a, b)\) it extends to a continuous function on closed bounded interval \([a, b]\) by Theorem 3.3.6. This extension is a bounded function by Theorem 3.2.1, hence \(f\) is also bounded.

Problem 5 (20 points). Prove that if \((f_n)\) is a sequence of uniformly continuous functions on a set \(D\) and if this sequence converges uniformly to the function \(f\) on \(D\), then \(f\) is also uniformly continuous.

Solution: Let \(\epsilon > 0\). Then there exists \(n_0\) such that \(n \geq n_0\) implies that \(|f_n(z) - f(z)| < \frac{\epsilon}{3}\) for all \(z \in D\). Pick some \(n \geq n_0\). Since \(f_n\) is uniformly continuous on \(D\), there exists \(\delta > 0\) such that \(|x - y| < \delta\) implies that \(|f(x) - f(y)| < \frac{\epsilon}{3}\). Therefore, we have
\[
|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|
\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]
for \(|x - y| < \delta\). It follows that \(f\) is uniformly continuous.