Math 3210-02: Homework 3, October 1, 2019

Solutions

Problem 1 (20 points)**.** Is the function $f(x) = \sin(\frac{1}{x})$ continuous on $(0,1)$? Is it uniformly continuous on $(0,1)$? Justify your answers.

Solution: Since $x \longmapsto \sin(x)$ is continuous on $\mathbb{R}$ and $x \longmapsto \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$, the function f is continuous on $(0,1)$.

Take sequences (x_n) and (y_n) given by $x_n = \frac{1}{\frac{\pi}{2}+2n\pi}$, and $y_n = \frac{1}{\frac{3\pi}{2}+2n\pi}$, in $(0,1)$. Then we have

$$0 \leq x_n - y_n = \frac{1}{\frac{\pi}{2} + 2n\pi} - \frac{1}{\frac{3\pi}{2} + 2n\pi} = \frac{\pi}{(\frac{\pi}{2} + 2n\pi)(\frac{3\pi}{2} + 2n\pi)} \leq \frac{1}{4n^2\pi}$$

and $|x_n - y_n| \to 0$ as $n \to \infty$.

On the other hand, we have

$$f(x_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

and

$$f(y_n) = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

Hence

$$|f(x_n) - f(y_n)| = 2,$$

and function f cannot be uniformly continuous.

Problem 2 (20 points)**.** Is the function $f(x) = x\sin(\frac{1}{x})$ uniformly continuous on $(0,1]$? Explain your answer!

Solution: As in the solution of Problem 1, we conclude that f is continuous on $(0,1]$. Moreover, we can extend it to $[0,1]$ by defining $f(0) = 0$. By the solution of Problem 3 in Homework 2 we know that this extended function is continuous on $[0,1]$. Therefore, f is uniformly continuous by Theorem 3.3.4.

Problem 3 (20 points)**.** Prove that the sequence $\frac{\sin(nx)}{n}$ converges uniformly to 0 on $[0,1]$.

Solution: We have

$$\left|\frac{\sin(nx)}{n}\right| \leq \frac{|\sin(nx)|}{n} \leq \frac{1}{n}$$

for all x. Let $\epsilon > 0$. Then there exists $N > 0$ such that $n > N$ implies that $\frac{1}{n} < \epsilon$ and

$$\left|\frac{\sin(nx)}{n}\right| < \epsilon$$

for all $x \in [0, 1]$.

Problem 4 (20 points)**.** Let $I = (a, b)$ be an open bounded interval. Let f be a uniformly continuous function on I. Show that f is bounded.

Solution: Since f is uniformly continuous on (a, b) it extends to a continuous function on closed bounded interval $[a, b]$ by Theorem 3.3.6. This extension is a bounded function by Theorem 3.2.1, hence f is also bounded.

Problem 5 (20 points)**.** Prove that if (f_n) is a sequence of uniformly continuous functions on a set D and if this sequence converges uniformly to the function f on D, then f is also uniformly continuous.

Solution: Let $\epsilon > 0$. Then there exists n_0 such that $n \geq n_0$ implies that $|f_n(z) - f(z)| < \frac{\epsilon}{3}$ for all $z \in D$. Pick some $n \geq n_0$. Since f_n is uniformly continuous on D, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \frac{\epsilon}{3}$. Therefore, we have

$$|f(x) - f(y)| = |(f(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - f(y))|$$
$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y))| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for $|x - y| < \delta$. It follows that f is uniformly continuous.