Problem 1 (20 points). Consider the sequence \((a_n)\) given by
\[ a_n = \frac{5 + (-1)^n n}{2 + 3n} \]
for \(n \in \mathbb{N}\).

(i) Does the sequence \((a_n)\) converge?

(ii) Does it have a convergent subsequence?

Problem 2 (20 points). Given a series \(\sum_{k=1}^{\infty} a_k\), define \(s_n = \sum_{k=1}^{n} a_k\) and \(t_n = \sum_{k=1}^{n} |a_k|\) for every \(n \in \mathbb{N}\). Prove that the sequence \((s_n)\) converges if the sequence \((t_n)\) is bounded. (Hint: Prove that \((s_n)\) is a Cauchy sequence!)

Problem 3 (20 points). Prove that the function \(f\) given by \(f(x) = x \sin \left(\frac{1}{x}\right)\) for \(x \neq 0\) and \(f(0) = 0\) is continuous at 0.

Problem 4 (20 points). Prove that a polynomial of odd degree has at least one real root. (Hint: Use Intermediate Value Theorem.)

Problem 5 (20 points). Let \(f\) be a continuous function on \([0, 1]\) with values in \([0, 1]\). Prove that there is a point \(c \in [0, 1]\) such that \(f(c) = c\) – that is show that \(f\) has a fixed point. (Hint: Apply the Intermediate Value Theorem to the function \(g(x) = f(x) - x\).)