Math 3220-02 Final, December 12, 2019

Solutions

Problem 1 (25 points)**.** Let x be an irrational number and r a rational number. Show that $x + r$ is also irrational.

Solution: Assume that $x + r$ is rational. Then $x = (x + r) - r$ would be the difference of two rational numbers. Therefore, it would be rational, contradicting our assumption.

Problem 2 (25 points)**.** Consder the function $f(x) = e^{-x^2}$ on real line. Show that it is uniformly continuous.

Solution: The derivative of f is

$$f'(x) = e^{-x^2} \cdot 2x = \frac{2x}{e^{x^2}}.$$

Since the exponential function e^{x^2} grows faster than x as $x \to +\infty$ we see that $\lim_{x\to+\infty} f'(x) = 0$ (which can also be checked using l'Hôspital rule). By the same reasoning, $\lim_{x\to-\infty} f'(x) = 0$. Therefore, there exists $M > 0$ such that $|f'(x)| < 1$ for $x < -M$ and $x > M$. On the other hand, f' is a continuous function, and therefore bounded on $[-M.M]$. Putting these facts together we see that f' is bounded. Then the assertion follows from Theorem 4.3.9.

Problem 3 (25 points)**.** Let f be a function differentiable on $[0, +\infty)$. Assume that $f(0) = 0$ and $|f'(x)| \leq M$. Prove that $|f(x)| \leq Mx$ on $[0, \infty)$. (Hint: use Mean Value Theorem!)

Solution: By the Mean Value Theorem we have

$$f(x) = f(x) - f(0) = f'(c)(x - 0) = f'(c)x$$

for some $0 < c < x$. Therefore, we have

$$|f(x)| = |f'(c)||x| \leq M|x| = Mx$$

since x is positive.

Problem 4 (25 points)**.** Give an example of a function f such that $|f|$ is integrable on $[0, 1]$ but f is not integrable on $[0, 1]$.

Solution: Let f be a function on $[0,1]$ such that $f(x) = 1$ for rational x, and $f(x) = -1$ for irrational x. It follows that, for any partition P of $[0,1]$, we have $L(f,P) = -1$ and $U(f,P) = 1$. Hence, f is not integrable. On the other hand, $|f|$ is equal to 1 on $[0,1]$ so it is integrable.

Problem 5 (25 points)**.** Let f is differentiable function on $[a,b]$ such that f' is integrable on $[a,b]$. Find

$$\int_a^b f(x)f'(x)\,dx$$

in terms of values of f. (Hint: Use the change of variables!)

Solution: Using the change of variables formula we have

$$\int_a^b f(x)f'(x)\,dx = \int_{f(a)}^{f(b)} u\,du = \left[\frac{u^2}{2}\right]_{f(b)}^{f(a)} = \frac{f(a)^2 - f(b)^2}{2}.$$

Problem 6 (25 points)**.** Find a power series on $(-1,1)$ which converges to

$$\frac{1}{(1-x^2)^2}.$$

Solution: The geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

converges for $|x| < 1$. By differentiation we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

By replacing x with x^2 we see that

$$\frac{1}{(1-x^2)^2} = \sum_{n=0}^{\infty}(n+1)x^{2n}$$

for $|x| < 1$.