Math 2270-1 Midterm 4, April 24, 2009

Solutions

Problem 1 (25 points)**.** Calculate the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(using the formula involving classical adjoint!).

Solution: By Sarrus rule we have

$$\det(A) = -1.$$

The classical adjoint is

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Hence, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem 2 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find

a) the eigenvalues of A;
b) the eigenvectors of A;
c) the algebraic and geometric multiplicity of these eigenvalues.

Does the matrix A have an eigenbasis? Is it similar to a diagonal matrix?

Solution: The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

Hence, the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

The algebraic multiplicity of 2 is equal to 1. The corresponding eigenvectors are proportional to

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The algebraic multiplicity of 1 is equal to 2. The corresponding eigenvectors are solutions of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

Hence, $c_2 = 0$ and $c_1 + c_3 = 0$. It follows that the eigenvectors are proportional to

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and the geometric multiplicity of 1 is equal to 1.

Therefore, A doesn't have an eigenbasis, and it is not similar to a diagonal matrix.

Problem 3 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the symmetric matrix

$$A = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

Find

a) the eigenvalues of A;
b) an orthonormal basis consisting of eigenvectors of A;
c) the "change of basis" matrix S from the standard basis to the basis above;
d) the matrix $D = S^{-1}AS$.

Solution: The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -2 & 2 \\ -2 & 1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = \lambda(1-\lambda)(1+\lambda)-4(1-\lambda)+4(1+\lambda)$$

$$= \lambda(1-\lambda)(1+\lambda) + 8\lambda = \lambda(9-\lambda^2) = -\lambda(\lambda-3)(\lambda+3).$$

Hence, the eigenvalues are $\lambda = 0$, $\lambda = 3$ and $\lambda = -3$. Their algebraic multiplicity is equal to 1.

For $\lambda = 0$, the eigenvectors satisfy the system

$$\begin{bmatrix} 0 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

From the first equation we see that $c_2 = c_3$. From the second we see that $c_1 = \frac{1}{2}c_2$. Hence the eigenvectors are proportional to

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

If we normalize the length to 1 we get

$$\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

For $\lambda = 3$, the eigenvectors satisfy the system

$$\begin{bmatrix} -3 & -2 & 2 \\ -2 & -2 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

From the second equation we see that $c_1 = -c_2$. From the third we see that $c_1 = 2c_3$. Hence the eigenvectors are proportional to

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

If we normalize the length to 1 we get

$$\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda = -3$, the eigenvectors satisfy the system

$$\begin{bmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

From the third equation we see that $c_1 = -c_3$. From the second we see that $c_1 = 2c_2$. Hence the eigenvectors are proportional to

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

If we normalize the length to 1 we get

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Hence the "change of basis" matrix is

$$S = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

We have

$$D = S^{-1}AS = S^T AS$$
$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 6 & -6 & 3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & -27 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Problem 4 (25 points)**.** Let $q(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 3x_2^2$ be a quadratic form. Find

a) the corresponding symmetric matrix A;
b) the eigenvalues of A;
b) an orthonormal basis consisting of eigenvectors of A;
c) the "change of basis" matrix S from the standard basis to the basis above;
d) the matrix $D = S^{-1}AS$.

Use this to diagonalize the quadratic form q.

Is q positive definite or not?

Solution: The matrix A is

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Its characteristic polynomial is

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)-1 = \lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4).$$

Hence, the eigenvalues are 2 and 4.

For eigenvalue 2, to find the eigenvectors we have to consider the linear system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0.$$

The second equation is the double of the first one, so we see that $c_1 = -c_2$. Hence we can pick an eigenvector of length 1 which is equal to

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For eigenvalue 4, to find the eigenvectors we have to consider the linear system

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0.$$

The second equation is the negative of the first one, so we see that $c_1 = c_2$. Hence we can pick an eigenvector of length 1 which is equal to

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It follows that

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned} D &= S^{-1}AS = S^T AS \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}. \end{aligned}$$

Hence, for $x_1 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$ and $x_2 = -\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$ we have

$$\begin{aligned} q(x_1, x_2) &= 3x_1^2 + 2x_1x_2 + 3x_2^2 \\ &= 3\left(\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2\right)^2 + 2\left(\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2\right)\left(-\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2\right) + 3\left(-\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2\right)^2 \\ &= 3\left(\frac{1}{2}y_1^2 + y_1y_2 + \frac{1}{2}y_2^2\right) + 2\left(-\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2\right) + 3\left(\frac{1}{2}y_1^2 - y_1y_2 + \frac{1}{2}y_2^2\right) \\ &= 2y_1^2 + 4y_2^2. \end{aligned}$$

Clearly, the form q is positive definite.